University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II
Professor: Hadi Salmasian

Second Midterm Exam – Version A

March 4, 2016

Surname ______________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ______________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	3	4	5	5	3	24
Grade							

1. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) **[2 points]** Calculate $A + 2A^T + I_3$.

Solution:

$$A + 2A^T + I_3 = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 2 \\ 6 & 4 & 0 \\ -2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ 6 & 7 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

(b) **[2 points]** Calculate AA^T.

Solution:

$$AA^T = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 7 & 0 \\ 7 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

2. [**3 points**] For each of the following statements, indicate if it is true (**T**) or false (**F**). You will receive .5 points for each correct answer and $-.5$ points for each incorrect answer (but you cannot receive a negative score on this question).

- T For every two $n \times n$ matrices A and B, we have: $(AB)^T = B^T A^T$.
- F For every two invertible $n \times n$ matrices A and B, we have: $(AB)^{-1} = A^{-1}B^{-1}$.
- T The columns of every invertible $n \times n$ matrix are linearly independent.
- F For every two $n \times n$ matrices A and B, if $AB = 0$ then $A = 0$ or $B = 0$.
- T For every positive integer n, the $n \times n$ identity matrix I_n is invertible.
- T If A is a 6×4 matrix, then the rows of A are always linearly dependent.

3. [**4 points**] Is the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix}$$

invertible? If the answer is yes, determine its inverse.

Solution: We row reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 0 & -2 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -3 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

We conclude that the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

4. An economy consists of two sectors: Industry and Service. In order to produce 1 unit of output, the Industry sector uses 0.2 units from the Industry and 0.6 units from the Service sector. Further, to produce 1 unit of output, the Service sector uses 0.5 units from Industry and 0.6 units from the Service sector.

(a) [**1 point**] Write down the consumption matrix C for this economy. **Solution:**

$$C = \begin{bmatrix} 0.2 & 0.5 \\ 0.6 & 0.6 \end{bmatrix}$$

(b) [**1 point**] Write down Leontief's input-output equation. **Solution:** $\vec{x} = C\vec{x} + \vec{d}$ or $(I_2 - C)\vec{x} = \vec{d}$.

(c) [**3 points**] Determine the production levels required to meet a final demand of 4 units from the Industry sector and 8 units from the Service sector.

Solution: We should solve the equation $A\vec{x} = \vec{d}$ where

$$A = I_2 - C = \begin{bmatrix} 0.8 & -0.5 \\ -0.6 & 0.4 \end{bmatrix}, \; \vec{d} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Now we have $detA = 0.8 \times 0.4 - 0.5 \times 0.6 = 0.32 - 0.3 = 0.02 \neq 0$, thus A is invertible and

$$A^{-1} = \frac{1}{0.02} \times \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.8 \end{bmatrix} = 50 \times \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 20 & 25 \\ 30 & 40 \end{bmatrix}$$

It follows that, $\vec{x} = A^{-1}d = \begin{bmatrix} 20 & 25 \\ 30 & 40 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 280 \\ 440 \end{bmatrix}$

5. [**5 points**] Consider the vectors

$$\vec{v_1} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \ \vec{v_2} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \ \vec{v_3} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \ \vec{v_4} = \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$$

Are the vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, and $\vec{v_4}$ linearly independent? If they are linearly dependent, then write down a linear dependence relation for these vectors. You should justify your answers.

Solution: We form the matrix $A = [\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}]$ and calculate the RREF:

$$A = \begin{bmatrix}1&1&1&2\\0&1&1&2\\1&1&0&2\\1&1&1&2\end{bmatrix} \xrightarrow[R_4\to R_4-R_1]{R_3\to R_3-R_1} \begin{bmatrix}1&1&1&2\\0&1&1&2\\0&0&-1&0\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow[R_1\to R_1+R_4]{R_2\to R_2+R_3} \begin{bmatrix}1&1&0&2\\0&1&0&2\\0&0&-1&0\\0&0&0&0\end{bmatrix} \xrightarrow[R_3\to -R3]{R_1\to R_1-R_2} \begin{bmatrix}1&0&0&0\\0&1&0&2\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

Since the fourth column is not a pivot column, the homogeneous linear system with coefficient matrix A has infinitely many solutions. Therefore the given vectors are linearly dependent.

To find the linear dependence relation, we should solve the homogeneous system with coefficient matrix A. The linear system corresponding to the RREF of A is:

$$\begin{cases} x_1 = 0 \\ x_2 + 2x_4 = 0 \\ x_3 = 0 \\ x_4 = \text{free} \end{cases}$$

Therefore the general solution of the system is given by:

$$\begin{cases} x_1 = 0 \\ x_2 = -2x_4 \\ x_3 = 0 \\ x_4 = \text{free.} \end{cases}$$

Now we can choose the value of x_4; e.g., $x_4 = 1$, and then we find a particular solution $x_1 = 0$, $x_2 = -2$, $x_3 = 0$, $x_4 = 1$, which results in the linear dependence relation $0\vec{v_1} - 2\vec{v_2} + 0\vec{v_3} + \vec{v_4} = \vec{0}$.

6. [**3 points**] Let A, B, and C be $n \times n$ invertible matrices. Find the $n \times n$ matrix X in terms of A, B, and C such that

$$BA(X+B)C^T - (CB)^T = 0$$

Solution:

$$BA(X+B)C^T - (CB)^T = 0 \Rightarrow BA(X+B)C^T = (CB)^T = B^T C^T$$
$$\Rightarrow X + B = (BA)^{-1}(B^T C^T)(C^T)^{-1} = A^{-1}B^{-1}B^T C^T (C^T)^{-1} = A^{-1}B^{-1}B^T$$
$$\Rightarrow X = A^{-1}B^{-1}B^T - B.$$