University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II
Professor: Hadi Salmasian

First Midterm Exam – Version A

February 5, 2016

Surname ______________________ First Name ______________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ______________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	3	3	2	3	19
Grade							

1. (a) [**3 points**] Find the general solution of the following linear equation. Write down the answer in vector parametric form.

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}.$$

(b) [**1 point**] Without using row reduction, find a solution of the vector equation

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

which satisfies $x_3 = -2$.

Hint: first use your solution for part (a) to write down the general solution for part (b).

Solution: (a) We row-reduce the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 1 & 0 & -2 \\ 1 & -3 & 2 & 6 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system corresponding to the REF is

$$\begin{cases} x_1 \quad\quad - x_3 = 0 \\ \quad\quad x_2 - x_3 = -2 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 - 2 \\ x_3 = \text{free} \end{cases}$$

Therefore the vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

(b) The coefficient matrix of the vector equation is identical to the coefficient matrix of the linear system in part (a). Thus, from the relation between the solution of a non-homogeneous system with its homogeneous form, we find that the general solution of the vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Setting $x_3 = -2$ yields $x_1 = x_2 = x_3 = -2$.

2. [**4 points**] For each of the following statements, determine if it is **true** or **false**.

For each correct answer, you will receive 1 point. For each incorrect answer, you will lose 1 point, but your total score for this question cannot be negative.

________ The number of pivot positions of the augmented matrix of a linear system is always equal to the number of its basic variables.

________ Every vector in the span of the columns of a 4×3 matrix A is of the form $A\mathbf{x}$ for some vector $\mathbf{x}$ in $\mathbb{R}^3$.

________ A homogeneous system can never be inconsistent.

________ The matrix $\begin{bmatrix} \frac{1}{2} & 2 & \frac{2}{3} & -\sqrt{2} & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form.

Solution: The answers from top to bottom are False, True, True, True.

3. Calculate the following.

(a) [**1 point**] $-\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} =$ **Solution:** $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

(b) [**2 points**] $\begin{bmatrix} 1 & -3 & \frac{1}{2} \\ 0 & 1 & -5 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} =$ **Solution:** $\begin{bmatrix} -3 \\ -9 \\ -3 \end{bmatrix}$

4. [**3 points**] Determine all values of h and k such that the linear system

$$\begin{cases} x & +y & -kz & = & 1 \\ hx & +2y & +2z & = & -1 \end{cases}$$

is inconsistent.

Solution: We reduce the augmented matrix to an echelon form:

$$\begin{bmatrix} 1 & 1 & -k & | & 1 \\ h & 2 & 2 & | & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - hR_1} \begin{bmatrix} 1 & 1 & -k & | & 1 \\ 0 & 2-h & 2+hk & | & -1-h \end{bmatrix}.$$

For the system to be inconsistent, we need the last row to be $[0 \ \ 0 \ \ 0 \ \ *]$ where $* \neq 0$. This implies that

$$2 - h = 0 \text{ and } 2 + hk = 0 \text{ and } -1 - h \neq 0.$$

From the first relation we obtain $h = 2$, and from the second one we then obtain $k = -1$. Finally, note that for $h = 2$ we have $-1 - h = -3 \neq 0$. Thus, exactly for the values

$$h = 2, k = -1$$

the system becomes inconsistent.

5. [**2 points**] Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ are vectors in $\mathbb{R}^m$ such that

$$\mathbf{u} = -\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3 \quad \text{and} \quad \mathbf{v} = 2\mathbf{w}_1 + 3\mathbf{w}_2 - \mathbf{w}_3.$$

Express $\mathbf{u}+2\mathbf{v}$ as a linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. In other words, find coefficients $c_1, c_2, c_3 \in \mathbb{R}$ such that $\mathbf{u} + 2\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$.

Solution: We can write

$$\begin{aligned}\mathbf{u} + 2\mathbf{v} &= (-\mathbf{w}_1 + \mathbf{w}_2 + 2\mathbf{w}_3) + 2(2\mathbf{w}_1 + 3\mathbf{w}_2 - \mathbf{w}_3) \\ &= (-\mathbf{w}_1 + 4\mathbf{w}_1) + (\mathbf{w}_2 + 6\mathbf{w}_2) + (2\mathbf{w}_3 - 2\mathbf{w}_3) \\ &= 3\mathbf{w}_1 + 7\mathbf{w}_2 = 3\mathbf{w}_1 + 7\mathbf{w}_2 + 0\mathbf{w}_3.\end{aligned}$$

Thus $c_1 = 3, c_2 = 7, c_3 = 0$.

6. [**3 points**] Consider the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ -2 \\ \frac{1}{2} \\ -2 \end{bmatrix}, \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -\frac{7}{2} \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -1 \\ -2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -\frac{1}{2} \\ -4 \end{bmatrix}$$

Determine if $\mathbf{u}$ belongs to Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: We reduce the augmented matrix $[\, \mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \mid \mathbf{u} \,]$ to an echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ -1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{7}{2} & -2 & -4 & -2 \end{array}\right] \xrightarrow[R_4 \to R_4 + \frac{7}{2}R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 5 & -\frac{1}{2} & -\frac{11}{2} \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -\frac{1}{2} & -\frac{11}{2} \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 5 & -\frac{1}{2} & -\frac{11}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in echelon form. Since it is the augmented matrix of a consistent linear system, it follows that $\mathbf{u}$ belongs to the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.