University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
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Final Exam – Solutions
April 2016

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 12 are *answer only*. For these questions, only your final answer will be considered for marks. Questions 13 through 20 are *long answer*. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) The number of points available for each question is indicated in square brackets.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.
(g) You may use the last page of the exam as scrap paper.
(h) Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with statement (h) above:

Signature: ______________________________

Good luck!

Please do not write in the table below.

Question	1–12	13	14	15	16	17	18	19	20	Total
Maximum	25	5	4	9	6	3	5	5	6	68
Grade										

Part A: Answer Only Questions

For Questions 1–12, only your final answer will be considered for marks. If applicable, write your final answers in the spaces provided.

1. **[2 points]** Consider the matrices

$$A = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ -\frac{1}{2} & -1 \end{bmatrix}.$$

Compute AB^T.

Answer: $AB^T = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}$.

2. **[2 points]** Let $z = 3i - 2$ and $w = 1 - 2i$. Write the complex number $\dfrac{|z-2|}{\bar{w}}$ in the form $a + bi$ where a and b are real numbers.

Answer: $\dfrac{|z-2|}{\bar{w}} = \dfrac{\sqrt{3^2 + (-4)^2}}{1+2i} = \dfrac{5}{1+2i} = \dfrac{5(1-2i)}{5} = 1 - 2i.$

3. **[2 points]** Let A, B, and C be 3×3 matrices such that $\det A = -1$, $\det B = \dfrac{1}{2}$, and $\det(C) = 4$. Calculate $\det(2A^{-1}C^TBAB^{-1}A^3)$.

Answer: $\det(C) = -32.$

4. **[2 points]** Determine all values of $k \in \mathbb{R}$ such that the linear system

$$\begin{cases} x + 3y + 2z = -1 \\ 2x - ky + 4z = 1 \end{cases}$$

is inconsistent. **Answer:** $k = -6$

5. **[2.5 points]** Let A be a $k \times \ell$ matrix, where $k < \ell$. For each statement below, write 'T' if the statement is true, and write 'F' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T A has at most k pivot columns.

F $\operatorname{rank} A + \dim \operatorname{Nul} A = k$.

F Every linear system of the form $A^T\mathbf{x} = \mathbf{b}$ is consistent.

F $AA^T = A^TA$.

T $\operatorname{Col}(A^T)$ is a subspace of $\mathbb{R}^\ell$.

6. **[2 points]** Let $A = \begin{bmatrix} 1 & 3 & -\frac{1}{3} \\ 0 & -2 & -\frac{1}{5} \\ 0 & 0 & -2 \end{bmatrix}$. Write down the eigenvalues of A^3 and their multiplicities.

Answer: -8 with multiplicity 2; 1 with multiplicity 1.

7. **[2 points]** Suppose that $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}$ and $W = \{\mathbf{x} \in \mathbb{R}^5 \mid A^T\mathbf{x} = \mathbf{0}\}$. Write down the dimension of W.

Answer: $W = \operatorname{Nul} A^T$, and so by the rank-nullity theorem $\dim W = 5 - 3 = 2$ because A^T is in echelon form.

8. **[2 points]** For each of the following subsets of $\mathbb{R}^3$, write 'Y' if the set is a subspace of $\mathbb{R}^3$ and write 'N' if it is not. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

Y $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & -7 \\ \frac{1}{2} & 6 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \text{ for some } s, t \in \mathbb{R} \right\}$

Y Span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right\}$

Y $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \right\}$

N $\left\{ \begin{bmatrix} s-t \\ s^2 \\ s+t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$

9. **[2 points]** Suppose that A is a square matrix and the characteristic equation of A is

$$\lambda^3 - 3\lambda^2 + 4 = 0.$$

For each of the following statements, write '**T**' if the statement is true, and write '**F**' if it is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

F $\lambda = 1$ is an eigenvalue of A.

T $\det(A) \neq 0$.

T $\text{Nul}(A + I) \neq \{\mathbf{0}\}$. (Here as usual "$I$" denotes the identity matrix.)

T The equation $A\mathbf{x} = 2\mathbf{x}$ has a solution other than $\mathbf{x} = \mathbf{0}$.

10. **[2.5 points]** For each statement below, write 'T' if the statement is true, and write 'F' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T A matrix can have more than one echelon form.

F The determinant of a square matrix is always equal to the determinant of its reduced echelon form.

T If $\mathbf{v}$ is a nonzero vector in $\mathbb{R}^n$, then $\{\mathbf{v}\}$ is a linearly independent set.

F A homogeneous linear system always has infinitely many solutions.

F Every 5 vectors in $\mathbb{R}^6$ are always linearly independent.

11. **[2 points]** Determine the value of $x \in \mathbb{R}$ such that the vector $\begin{bmatrix} x \\ i-1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = i$ for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{i} \end{bmatrix}.$$

Answer: $x = 1$, so the vector is $\begin{bmatrix} 1 \\ i-1 \end{bmatrix}$.

12. **[2 points]** Let A and B be $n \times n$ invertible matrices. Solve the matrix equation $(X^T A + B^T)^T = A^T A$ for the matrix X.

Answer: $X = A - (A^T)^{-1} B$.

Part B: Long Answer Questions

For Questions 13–20, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

13. [**5 points**] Is the following linear system consistent or inconsistent? If it is consistent, then write down the general solution in vector parametric form.

$$\begin{cases} x_1 + x_3 &= 3x_2 + x_4 + x_5 \\ x_3 + x_5 &= 2x_4 + 1 \\ 3x_2 + 3x_4 &= x_1 + 2x_3 - 1 \end{cases}$$

Solution: We reduce the augmented matrix of the system to the REF:

$$\begin{bmatrix} 1 & -3 & 1 & -1 & -1 & | & 0 \\ 0 & 0 & 1 & -2 & 1 & | & 1 \\ 1 & -3 & 2 & -3 & 0 & | & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & -3 & 1 & -1 & -1 & | & 0 \\ 0 & 0 & 1 & -2 & 1 & | & 1 \\ 0 & 0 & 1 & -2 & 1 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -3 & 1 & -1 & -1 & | & 0 \\ 0 & 0 & 1 & -2 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & -3 & 0 & 1 & -2 & | & -1 \\ 0 & 0 & 1 & -2 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

It follows that the system is consistent (since the rightmost column is not pivot). Moreover, x_1, x_3 are basic, x_2, x_4, x_5 are free. The general solution is

$$\begin{cases} x_1 = 3x_2 - x_4 + 2x_5 - 1 \\ x_2 = \text{free} \\ x_3 = 2x_4 - x_5 + 1 \\ x_4 = \text{free} \\ x_5 = \text{free} \end{cases}$$

The vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - x_4 + 2x_5 - 1 \\ x_2 \\ 2x_4 - x_5 + 1 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

14. **[4 points]** Calculate the determinant of

$$M = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 7 & 1 & -1 \\ 1 & 15 & 2 & 1 \end{bmatrix}.$$

Solution: Expanding with respect to the second row, and then with respect to the first column of the 3×3 matrix, we obtain

$$\begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 7 & 1 & -1 \\ 1 & 15 & 2 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (-1) \left((1) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - (0) \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + (1) \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \right)$$

$$= (-1)(3 + (-5)) = 2.$$

15. Consider the matrix $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ \frac{1}{2} & 0 & 2 \end{bmatrix}$.

(a) [**3 points**] Find the eigenvalues of B.

Solution: Starting with an expansion with respect to the second column, we have

$$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 1 & 1-\lambda & 2 \\ \frac{1}{2} & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-4\lambda+3) = -(\lambda-1)^2(\lambda-3).$$

Thus, the eigenvalues are 1 (with multiplicity two) and 3 (with multiplicity one).

(b) [**4 points**] For each of the eigenvalues of B found in part (a), find a basis of the corresponding eigenspace.

Solution: For the eigenvalue $\lambda = 1$, we have:

$$\left[\ B-I \mid 0\ \right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For the eigenvalue $\lambda = 3$, we have:

$$\left[\ B-3I \mid 0\ \right] = \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + \frac{1}{2}R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to -R_1]{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(c) **[2 points]** Find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$. You do *not* need to calculate P^{-1}.

Solution: Since the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue, the matrix B is diagonalizable. If

$$P = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then we have $B = PDP^{-1}$.

16. Latin dance shoes are hard to find in Hungary. There are only two brands available: *Esbrezzo* and *Litheslide*, each of which has an equal share of the market. After three months of marketing competition, 40% of Esbrezzo's regular customers switch to Litheslide, while 30% of the Litheslide customers switch to Esbrezzo.

(a) **[1 point]** Write down the migration matrix M and the initial state vector $\vec{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Another possible migration matrix (if one reverses the order of the companies) is

$$M = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}.$$

(b) **[1 point]** Write down the market share of each of the companies after three months.

Solution: $\vec{x}_1 = M\vec{x}_0 = \begin{bmatrix} .45 \\ .55 \end{bmatrix}$

(c) **[4 points]** If the same marketing campaign continues for several more months, in the long run what is the predicted market share of each company?

Solution: To find the steady-state vector, we must find an eigenvector of eigenvalue 1. We row reduce:

$$\left[\begin{array}{c|c} M - I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.4 & 0.3 & 0 \\ 0.4 & -0.3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} -0.4 & 0.3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{5}{2} R_1} \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is thus

$$\vec{x} = x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}.$$

We choose the value of the free variable so that the sum of the entries of $\vec{x}$ is equal to one:

$$\left(\frac{3}{4} + 1\right) x_2 = 1 \implies x_2 = \frac{4}{7}.$$

Thus the steady-state vector is

$$\vec{x} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}.$$

Since M is regular stochastic, the long term behaviour is given by the steady-state vector. Thus, in the long term, Esbrezzo will hold 3/7 of the market and Litheslide will hold 4/7 of the market.

17. [**3 points**] Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 & -4 \\ 3 & 0 & 2 & -2 & -4 \\ 0 & 0 & 0 & -4 & -2 \\ -3 & -1 & -1 & 2 & 5 \end{bmatrix}.$$

Find a basis for Col A.

Solution: We row reduce to find an echelon form of A.

$$A \xrightarrow[R_4 \to R_4 + R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + \frac{1}{2}R_3} \begin{bmatrix} 3 & 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the pivot columns are columns 1, 2, and 4. Thus, a basis of Col A is

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \\ 2 \end{bmatrix} \right\}.$$

18. Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

(a) **[4 points]** Find the inverse of A.

Solution: We row reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 6 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 6R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -6 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 0 & 1 & -2 & -6 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 5 & 12 & -2 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 0 & 1 & -2 & -6 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -2 & -5 & 1 \\ 0 & 0 & 1 & -2 & -6 & 1 \end{array}\right]$$

Thus A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -5 & 1 \\ -2 & -6 & 1 \end{bmatrix}.$$

(b) **[1 point]** Using the result of part (a), find a row vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ such that

$$\mathbf{x}A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Solution: We have

$$\mathbf{x}A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \Rightarrow \mathbf{x} = \mathbf{x}AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -3 & -9 & 2 \end{bmatrix}.$$

19. [**5 points**] Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 5 \\ -1 \\ 2 \\ 8 \end{bmatrix}.$$

Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ linearly independent? If not, find a linear dependence relation.

Solution: We form the matrix whose columns are the given vectors and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 8 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 + R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

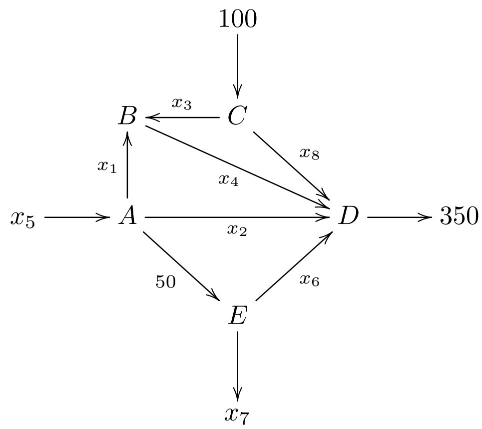
Since there is a free variable, the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are linearly dependent. To find a linear dependence relation, we find a nontrivial solution to the corresponding homogeneous system. The general solution is

$$\vec{x} = x_4 \begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \qquad x_4 \in \mathbb{R}.$$

We choose any nonzero value for x_4. For instance, if we take $x_4 = 1$, we obtain the linear dependence relation

$$-3\vec{v}_1 + \vec{v}_2 - 2\vec{v}_3 + \vec{v}_4 = 0.$$

20. Consider the traffic flow described by the following diagram. The letters A through E label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



(a) **[3 points]** Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 8$. (Do not solve the linear system.)

Solution:

A	x_5	$= x_1 + x_2 + 50$
B	$x_1 + x_3$	$= x_4$
C	100	$= x_3 + x_8$
D	$x_2 + x_4 + x_6 + x_8$	$= 350$
E	50	$= x_6 + x_7$
Total	$x_5 + 100$	$= x_7 + 350$

(b) **[3 points]** The reduced echelon form of the linear system from part (a) is as follows:

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 300 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 250 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down the general flow pattern. Then determine the maximum possible value of x_7 (you should justify your answer).

Solution: The general flow patter is:

$$x_1 = x_4 + x_8 - 100$$
$$x_2 = -x_4 + x_7 - x_8 + 300$$
$$x_3 = -x_8 + 100$$
$$x_4 \text{ free}$$
$$x_5 = x_7 + 250$$
$$x_6 = -x_7 + 50$$
$$x_7 \text{ free}$$
$$x_8 \text{ free}$$

Since all the x_i's should be ≥ 0, from $x_6 \geq 0$ it follows that $x_7 \leq 50$. So the maximum value of x_7 is 50.