University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II
Professor: Alistair Savage

Third Midterm Test – Blue Version – Solutions
27 March 2015

Surname ________________________ First Name ________________________

Student # ______________________ DGD (1–4) ________________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	Total
Maximum	5	4	4	4	3	3	5	28
Grade								

QUESTION 1. [**5 points**] Find a basis of the null space of the matrix

$$B = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & -2 \\ 0 & 1 & -2 & -1 & 3 & -5 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}.$$

What is rank B?

Solution: We row reduce to RREF:

$$\begin{bmatrix} 1 & 1 & 2 & 0 & -1 & -2 \\ 0 & 1 & -2 & -1 & 3 & -5 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & -2 \\ 0 & 1 & -2 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & 4 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

The reduced matrix corresponds to the linear system

$$\begin{array}{ccccccccccc} x_1 & & + & 4x_3 & & - & 3x_5 & & & = & 0 \\ & x_2 & - & 2x_3 & & + & 2x_5 & - & 2x_6 & = & 0 \\ & & & & x_4 & - & x_5 & + & 3x_6 & = & 0 \end{array}$$

The general solution to the linear system is therefore:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= 2x_3 - 2x_5 + 2x_6 \\ x_4 &= x_5 - 3x_6 \end{aligned}$$

The solution set in vector parametric form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \qquad x_3, x_5, x_6 \in \mathbb{R}.$$

Therefore, a basis for Nul A is

$$\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The matrix B has 3 pivot columns. Thus, rank $B = 3$.

Question 2. [**4 points**] For each of the following sets, write 'Y' if the set is a subspace of $\mathbb{R}^n$ for the given n and write 'N' if the set is not a subspace of $\mathbb{R}^n$ for the given n. You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

Y The set of solutions to a homogeneous linear system in n variables. Here n can be any positive integer.

Y The set $\{(x-3y, 2y-z, x+2y+z) \mid x, y, z \in \mathbb{R}\}$, $n = 3$.

N A line in $\mathbb{R}^2$ that does not pass through the origin, $n = 2$.

Y The span of 5 vectors in $\mathbb{R}^4$, $n = 4$.

N The set $\{(x, 1, x+y, y) \mid x, y \in \mathbb{R}\}$, $n = 4$.

Y The null space of a 3×5 matrix, $n = 5$.

N The column space of a 7×8 matrix, $n = 8$.

Y A plane in $\mathbb{R}^3$ containing the origin, $n = 3$.

Question 3. [**4 points**] For each statement below, indicate if it is true (T) or false (F). You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T For every $n \times m$ matrix A, the equality $\operatorname{rank} A + \dim \operatorname{Nul} A = m$ holds.

T If all of the entries of an $n \times n$ matrix A are integers, then the determinant of A is an integer.

F The set $\{\vec{0}\}$ is a basis of the subspace $\{\vec{0}\}$.

T Every nonconstant polynomial has a complex root.

F The product of two imaginary numbers is an imaginary number.

F If V is a subspace of $\mathbb{R}^n$, then every basis of V has n elements.

T A subspace can have more than one basis.

F A set of 3 vectors in $\mathbb{R}^5$ is always linearly independent.

Question 4. [**4 points**] Compute the determinant of the matrix

$$A = \begin{bmatrix} -3 & 0 & -2 & 0 & -1 \\ 10 & 4 & 17 & 25 & -18 \\ 5 & 0 & 3 & 0 & 2 \\ 3 & 0 & -10 & -1 & 8 \\ -1 & 0 & 2 & 0 & 0 \end{bmatrix}$$

Solution: We first expand along the second column:

$$\det A = 4 \begin{vmatrix} -3 & -2 & 0 & -1 \\ 5 & 3 & 0 & 2 \\ 3 & -10 & -1 & 8 \\ -1 & 2 & 0 & 0 \end{vmatrix}$$

Next, we expand along the third column:

$$\det A = (4)(-1) \begin{vmatrix} -3 & -2 & -1 \\ 5 & 3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

Now we expand along the third row (the third column would also be a good choice):

$$\det A = (-4)\left(-1\begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix}\right)$$

$$= (-4)\Big(-1\big((-2)2-(-1)3\big) - 2\big((-3)2-(-1)5\big)\Big) = (-4)\big(-(-1)-2(-1)\big) = -12$$

QUESTION 5. **[3 points]** Suppose $z = 2 - 3i$ and $w = -3 - 4i$. Calculate

$$\bar{z}w \quad \text{and} \quad \frac{z}{w}.$$

Write your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Solution: We have

$$\bar{z}w = \overline{(2-3i)}(-3-4i) = (2+3i)(-3-4i) = -6 - 8i - 9i + 12 = 6 - 17i$$

and

$$\frac{z}{w} = (2-3i) \cdot \frac{-3+4i}{9+16} = \frac{-6+8i+9i+12}{25} = \frac{6}{25} + \frac{17}{25}i.$$

QUESTION 6. Suppose A, B, and C are invertible 3×3 matrices with $\det A = 2$ and $\det C = -2$. Furthermore, suppose that

$$X = 2B^TCA^2B^{-1}A^{-1}.$$

(a) **[2 points]** What is $\det X$?

Solution: We have

$$\det X = \det(2B^TCA^2B^{-1}A^{-1}) = 2^3(\det B^T)(\det C)(\det A)^2(\det B^{-1})(\det A)^{-1}$$
$$= 8 \cdot (\det B) \cdot (-2) \cdot 4 \cdot (\det B)^{-1} \cdot \frac{1}{2} = -32$$

(b) **[1 point]** Is X invertible?

Solution: Yes, X is invertible because its determinant is nonzero.

QUESTION 7. [**5 points**] Find a basis for

$$\text{Span}\left\{\begin{bmatrix}2\\2\\0\\0\\-2\end{bmatrix},\begin{bmatrix}-4\\-4\\0\\0\\4\end{bmatrix},\begin{bmatrix}-1\\4\\5\\0\\1\end{bmatrix},\begin{bmatrix}4\\4\\2\\4\\-4\end{bmatrix},\begin{bmatrix}3\\5\\10\\16\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\-1\\4\\5\end{bmatrix}\right\}$$

What is the dimension of this span?

Solution: We form the matrix with the given vectors as columns and row reduce to find the pivot columns:

$$\begin{bmatrix}2&-4&-1&4&3&-1\\2&-4&4&4&5&-3\\0&0&5&2&10&-1\\0&0&0&4&16&4\\-2&4&1&-4&-3&5\end{bmatrix}\xrightarrow[R_1+R_5]{-R_1+R_2}\begin{bmatrix}2&-4&-1&4&3&-1\\0&0&5&0&2&-2\\0&0&5&2&10&-1\\0&0&0&4&16&4\\0&0&0&0&0&4\end{bmatrix}$$

$$\xrightarrow{-R_2+R_3}\begin{bmatrix}2&-4&-1&4&3&-1\\0&0&5&0&2&-2\\0&0&0&2&8&1\\0&0&0&4&16&4\\0&0&0&0&0&4\end{bmatrix}\xrightarrow{-2R_3+R_4}\begin{bmatrix}2&-4&-1&4&3&-1\\0&0&5&0&2&-2\\0&0&0&2&8&1\\0&0&0&0&0&2\\0&0&0&0&0&4\end{bmatrix}$$

$$\xrightarrow{-2R_4+R_5}\begin{bmatrix}2&-4&-1&4&3&-1\\0&0&5&0&2&-2\\0&0&0&2&8&1\\0&0&0&0&0&2\\0&0&0&0&0&0\end{bmatrix}$$

We see that the first, third, fourth, and sixth columns are pivot columns. Thus, a basis for the given span is

$$\left\{\begin{bmatrix}2\\2\\0\\0\\-2\end{bmatrix},\begin{bmatrix}-1\\4\\5\\0\\1\end{bmatrix},\begin{bmatrix}4\\4\\2\\4\\-4\end{bmatrix},\begin{bmatrix}-1\\-3\\-1\\4\\5\end{bmatrix}\right\}.$$

Therefore, the dimension of the span is 4.