University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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Second Midterm Test – Blue Version – Solutions

6 March 2015

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	6	5	6	6	6	33
Grade							

QUESTION 1. [**4 points**] For each statement below, indicate if it is true (T) or false (F). You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

T For all $n \times n$ matrices A and B, if A and B are invertible, then the product AB is invertible.

F For all $n \times n$ matrices A and B, if A and B are invertible, then the sum $A + B$ is invertible.

F For all $n \times n$ matrices A and B, it is true that $(AB)^T = A^T B^T$.

T For all $n \times n$ matrices A and B, it is true that $(A + B)^T = A^T + B^T$.

T The identity matrix I_6 is invertible.

F If A is an $n \times n$ invertible matrix, then there exists a vector $\vec{b} \in \mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

T The matrix $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ is singular.

F If A is an $n \times n$ matrix such that the equation $A\vec{x} = \vec{0}$ has a nontrivial solution, then A is invertible.

QUESTION 2.

(a) [**2 points**] Let

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Calculate B^2.

Solution:

$$B^2 = \begin{bmatrix} 3 & 3 & -1 \\ -3 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) [**4 points**] Let

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Find the matrix X satisfying the equation

$$(3X^T + A)B = I_2.$$

Here I_2 is the 2×2 identity matrix.

Solution: The matrix B is invertible and

$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned}
(3X^T + A)B = I_2 &\implies 3X^T + A = I_2 B^{-1} = B^{-1} \\
&\implies 3X^T = B^{-1} - A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} \\
&\implies X^T = \frac{1}{3} \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \\
&\implies X = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}.
\end{aligned}$$

QUESTION 3. An economy consists of two sectors: industry and agriculture. In order to produce one unit, the industry sector consumes 0.4 units from industry and 0.8 units from agriculture. On the other hand, in order to produce one unit, the agriculture sector consumes 0.5 units from industry and 0.3 units from agriculture.

(a) [**1 point**] Write the consumption matrix C for this economy.

Solution:

$$C = \begin{bmatrix} 0.4 & 0.5 \\ 0.8 & 0.3 \end{bmatrix}$$

(b) [**1 point**] State the Leontief Input-Output Model production equation.

Solution: The equation is $\vec{x} = C\vec{x} + \vec{d}$, or $(I - C)\vec{x} = \vec{d}$.

(c) [**3 points**] Determine the production levels necessary to satisfy a final demand of 10 units from industry and 20 units from agriculture.

Solution: We solve the production equation $A\vec{x} = \vec{d}$, where

$$A = I_2 - C = \begin{bmatrix} 0.6 & -0.5 \\ -0.8 & 0.7 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

We have $\det A = 0.6 \cdot 0.7 - 0.8 \cdot 0.5 = 0.42 - 0.4 = 0.02 \neq 0$, so A is invertible and

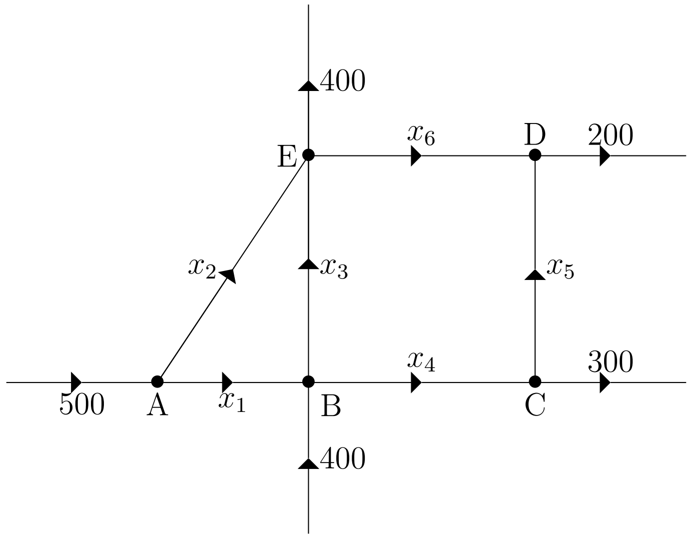
$$A^{-1} = \frac{1}{0.02} \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.6 \end{bmatrix} = 50 \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.6 \end{bmatrix} = \begin{bmatrix} 35 & 25 \\ 40 & 30 \end{bmatrix}.$$

Thus $\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 35 & 25 \\ 40 & 30 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 850 \\ 1000 \end{bmatrix}$. So industry must produce 850 units and agriculture must produce 1000 units.

Note: The system can also be solved by row reduction.

QUESTION 4.

(a) [**3 points**] Write a system of linear equations describing the flow in the following network. The letters A through E label nodes in the network and the arrows indicate the direction of flow. You do *not* need to solve the system.



Solution: Setting the total flow in equal to the total flow out at each vertex, we get the following linear system.

$$\begin{aligned}
A:&\quad 500 &&= x_1 + x_2,\\
B:&\quad x_1 + 400 &&= x_3 + x_4,\\
C:&\quad x_4 &&= x_5 + 300,\\
D:&\quad x_5 + x_6 &&= 200,\\
E:&\quad x_2 + x_3 &&= x_6 + 400,\\
\text{Overall:}&\quad 500 + 400 &&= 400 + 200 + 300
\end{aligned}$$

The simplified system is:

$$\begin{aligned}
x_1 + x_2 &= 500\\
-x_1 + x_3 + x_4 &= 400\\
x_4 - x_5 &= 300\\
x_5 + x_6 &= 200\\
x_2 + x_3 - x_6 &= 400
\end{aligned}$$

where we have omitted the last equation, which becomes $0 = 0$.

(b) [**3 points**] The reduced echelon form of the augmented matrix associated to the linear system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & 0 & -1 & 400 \\ 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Find the general solution to this system. What are the minimum and maximum values for x_4 and x_6? (Use the fact that the flow along each edge in the network must be nonnegative.)

Solution: The general solution to the system is:

$$\begin{aligned} x_1 &= 100 + x_3 - x_6 \\ x_2 &= 400 - x_3 + x_6 \\ x_4 &= 500 - x_6 \\ x_5 &= 200 - x_6 \\ x_3, x_6 &\text{ free, but } \geq 0 \end{aligned}$$

Since $x_5 \geq 0$, we have $0 \leq x_6 \leq 200$ (i.e. the minimum value for x_6 is zero and the maximum value is 200.) Thus, from the expression for x_4, we have $300 \leq x_4 \leq 500$ (i.e. the minimum value for x_4 is 300 and the maximum value is 500).

Question 5. [**6 points**] Are the following sets linearly independent? Justify your answers.

(a)

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

Solution: No, there are more vectors than entries in each vector (3 vectors in $\mathbb{R}^2$ and $3 > 2$).

(b)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution: Yes, because there are two vectors in the set, and they are not parallel (as can be seen, for instance, by looking at the last coordinates).

(c)

$$\left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Solution: Yes, since the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

with these vectors as columns has a pivot in each column. Thus, the homogeneous system with this coefficient matrix has only the trivial solution.

QUESTION 6.

(a) [**4 points**] Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

invertible? If so, find its inverse.

Solution: We row reduce the superaugmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 9 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-4R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 9 & -4 & 0 \\ 0 & 1 & 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

Thus A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

(b) **[2 points]** Let A be the matrix from part (a). Solve the equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is the vector

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: Since A is invertible,

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$