University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II
Professor: Alistair Savage

First Midterm Exam – Blue Version

February 6, 2015

Surname ______________________ First Name ____________________

Student # ____________________ DGD ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	3	5	3	3	22
Grade							

1. (a) [**3 points**] Find the general solution of the following vector equation. Write the answer in *vector parametric form.*

$$x_1 \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 6 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -6 \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) [**1 point**] The values

$$x_1 = -7, \ x_2 = 10, \ \text{and } x_3 = 15$$

form a solution of the linear system

$$\begin{bmatrix} 3 & 6 & -5 \\ 3 & 6 & -6 \\ -1 & -2 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -36 \\ -51 \\ 12 \end{bmatrix}.$$

Without using row reduction, find the general solution for the above linear system. Write your answer in *vector parametric form.*

Hint. Compare the coefficient matrix of the linear system in part (b) with the one in part (a).

Solution: (a) We row reduce the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & 6 & -5 & 0 \\ 3 & 6 & -6 & 0 \\ -1 & -2 & \frac{5}{3} & 0 \end{array}\right] \xrightarrow[\frac{1}{3}R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 3 & 6 & -5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 3 & 6 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{5R_2+R_1} \left[\begin{array}{ccc|c} 3 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system corresponding to the RREF is

$$\begin{array}{cccc} x_1 & +2x_2 & & = 0 \\ & & x_3 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = -2x_2 \\ x_2 \text{ free} \\ x_3 = 0 \end{array}$$

Therefore the vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R}.$$

(b) The system in part (a) is simply the homogeneous system corresponding to the system in part (b). Therefore, the solution set to the system in part (b) is obtained from the solution set in part (a) by adding a particular solution. Thus, the general solution to the system in part (b) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \\ 15 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R}.$$

2. [**4 points**] Determine if the following linear system is consistent or inconsistent. (If the system is consistent, you do *not* need to find the general solution.)

$$\begin{aligned} 5 + 2x_1 + x_2 &= x_3 + 4x_4 + x_5 \\ 2x_2 + x_3 + 3x_4 + 3x_5 &= 2x_1 + 6 \\ 2x_1 - x_3 &= 2x_4 + 5x_5 + 5x_2 + 10 \end{aligned}$$

Solution: The linear system in standard form is:

$$\begin{array}{rrrrrcr} 2x_1 & +x_2 & -x_3 & -4x_4 & -x_5 & = & -5 \\ -2x_1 & +2x_2 & +x_3 & +3x_4 & +3x_5 & = & 6 \\ 2x_1 & -5x_2 & -x_3 & -2x_4 & -5x_5 & = & 10 \end{array}$$

We row reduce the augmented matrix to an echelon form:

$$\left[\begin{array}{ccccc|c} 2 & 1 & -1 & -4 & -1 & -5 \\ -2 & 2 & 1 & 3 & 3 & 6 \\ 2 & -5 & -1 & -2 & -5 & 10 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccccc|c} 2 & 1 & -1 & -4 & -1 & -5 \\ 0 & 3 & 0 & -1 & 2 & 1 \\ 2 & -5 & -1 & -2 & -5 & 10 \end{array}\right]$$

$$\xrightarrow{-R_1+R_3} \left[\begin{array}{ccccc|c} 2 & 1 & -1 & -4 & -1 & -5 \\ 0 & 3 & 0 & -1 & 2 & 1 \\ 0 & -6 & 0 & 2 & -4 & 15 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccccc|c} 2 & 1 & -1 & -4 & -1 & -5 \\ 0 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 17 \end{array}\right]$$

Since the rightmost column is a pivot column, the linear system is inconsistent.

3. [**3 points**] For each of the following statements, indicate if it is *true* or *false*. You will receive 1 point for every correct answer and lose 0.5 points for every incorrect answer (but you cannot receive a negative mark on this question).

(a) A consistent linear system with 6 equations and 6 variables always has a unique solution.

(b) If the span of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ is a line, then there exists a scalar $c \in \mathbb{R}$ such that $\vec{v} = c\vec{w}$ or $\vec{v} = c\vec{v}$.

(c) If A is a 3×5 matrix and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, then the homogeneous linear system $A\vec{x} = \vec{0}$ has infinitely many solutions.

Solution: (a) False
(b) True
(c) True

4. (a) **[3 points]** For which value(s) of t does the following linear system have infinitely many solutions?

$$\begin{array}{cccl} x_1 & +x_2 & -x_3 & = 1 \\ 2x_1 & +x_2 & -3x_3 & = -3 \\ x_1 & +x_2 & +tx_3 & = 1 \end{array}$$

Solution: We row reduce the augmented matrix to an echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 1 & -3 & -3 \\ 1 & 1 & t & 1 \end{array}\right] \xrightarrow[-R_1+R_2]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & t+1 & 0 \end{array}\right]$$

The system has infinitely many solutions exactly when it is consistent and has at least one free variable. These conditions are satisfied if and only if $t = -1$.

(b) **[2 points]** Write down the general solution of the linear system for the value of t that you find in part (a).

Solution: For $t = -1$ we row reduce the augmented matrix to reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system corresponding to the REF is

$$\begin{array}{cccl} x_1 & & -2x_3 & = -4 \\ & x_2 & +x_3 & = 5 \end{array} \quad \Rightarrow \quad \begin{array}{ll} x_1 & = 2x_3 - 4 \\ x_2 & = -x_3 + 5 \\ x_3 & \text{free} \end{array}$$

5. [**3 points**] Consider the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -\frac{1}{2} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \\ \frac{1}{2} \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 7 \\ -4 \\ 2 \end{bmatrix}.$$

Does $\vec{u}$ belong to $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

Solution: We form a matrix with the above vectors as its columns and row reduce:

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ -1 & 4 & 7 & 4 \\ 2 & -2 & -4 & 4 \\ -\frac{1}{2} & \frac{1}{2} & 2 & -1 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1+R_4]{\substack{R_1+R_2 \\ -2R_1+R_3}} \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 3 & 5 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rightmost column is not a pivot column, it follows that $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

6. Calculate the following.

(a) [**1 point**] $3\begin{bmatrix}2\\1\\2\end{bmatrix} - 2\begin{bmatrix}-2\\3\\5\end{bmatrix}$

Solution: $\begin{bmatrix}10\\-3\\-4\end{bmatrix}$

(b) [**2 points**] $\begin{bmatrix}1 & 0 & -3\\-1 & 2 & 1\\1 & 2 & -2\end{bmatrix}\begin{bmatrix}-1\\-2\\1\end{bmatrix}$

Solution: $\begin{bmatrix}-4\\-2\\-7\end{bmatrix}$