University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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Final Exam – Solutions

April 2015

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 13 are *answer only*. For these questions, only your final answer will be considered for marks. Questions 14 through 21 are *long answer*. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(g) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.

(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–13	14	15	16	17	18	19	20	21	Total
Maximum	30	4	5	9	5	5	6	6	5	75
Grade										

Part A: Answer Only Questions

For Questions 1–13, only your final answer will be considered for marks. If applicable, write your final answers in the spaces provided.

1. [**3 points**] Consider the matrices

$$A = \begin{bmatrix} -1 & 2 \\ 4 & -2 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & \frac{1}{3} \\ 0 & 1 \end{bmatrix}.$$

Compute BA^T and $B - 2C$.

Answer: $BA^T = \begin{bmatrix} 3 & -3 & 9 \\ -3 & 6 & -10 \end{bmatrix}$ and $B - 2C = \begin{bmatrix} 4 & \frac{5}{6} \\ 1 & -3 \end{bmatrix}$.

2. [**2 points**] Let $z = 1 + i$ and $w = 2 - i$. Write the complex numbers $(z+2)\bar{w}$ and $\dfrac{z}{w}$ in the form $a + bi$ where a and b are real numbers.

Answer: $(z+2)\bar{w} = 5 + 5i$, and $\dfrac{z}{w} = \dfrac{1}{5} + \dfrac{3}{5}i$.

3. [**2 points**] Suppose A, B, and C are 3×4 matrices and $\vec{v} \in \mathbb{R}^4$ such that

$$C = 2A - B, \quad A\vec{v} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \quad \text{and} \quad B\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}.$$

What is $C\vec{v}$?

Answer: $C\vec{v} = (2A - B)\vec{v} = 2A\vec{v} - B\vec{v} = 2\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix}$.

4. [**2 points**] Let

$$A = \begin{bmatrix} 1+i & 3-i & -\frac{1}{2}+i & 4 \\ 0 & i & -\frac{1}{1+i} & -3 \\ 0 & 0 & \frac{1}{i} & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Write down the eigenvalues of A^2 and their multiplicities.

Answer: $(1+i)^2 = 2i$ with multiplicity one, -1 with multiplicity two, and 1 with multiplicity one.

5. [**2 points**] Let A, B, and C be 3×3 matrices such that $\det A = -1$, $\det B = \frac{1}{2}$, and $\det(2AB^TCA^{-2}) = 1$. Calculate $\det C$.

Answer: $\det(C) = -\frac{1}{4}$.

6. [**3 points**] Suppose n is an integer greater than or equal to two. For each condition below, write 'Y' if the condition implies that an $n \times n$ matrix A is invertible, and write 'N' if the condition does not imply that an $n \times n$ matrix A is invertible. You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

Y Every vector in $\mathbb{R}^n$ can be written as a linear combination of columns of A.

N The homogeneous linear system $A\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$.

N $\operatorname{Nul} A = \mathbb{R}^n$.

Y $\operatorname{rank} A = n$.

Y The characteristic equation of A is $\lambda^n - 2\lambda^{n-1} + 5$.

N $\lambda = 1$ is an eigenvalue of A.

7. [**3 points**] For each statement below, write 'T' if the statement is true, and write 'F' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

F If A is an $n \times n$ matrix with $\operatorname{rank} A < n$, then the homogeneous system $A\vec{x} = \vec{0}$ is inconsistent.

F Every elementary row operation leaves the determinant of every square matrix unchanged.

T If the columns of a matrix A are linearly dependent, then the homogeneous system $A\vec{x} = \vec{0}$ has infinitely many solutions.

F The scalar zero cannot be an eigenvalue of a matrix.

T If A is a square matrix such that $\det A = 0$, then the rows of A are linearly dependent.

T A matrix is invertible if and only if its determinant is nonzero.

8. [**2 points**] For each of the following subsets of $\mathbb{R}^3$, write 'Y' if the set is a subspace of $\mathbb{R}^3$ and write 'N' if it is not. You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

Y $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \text{ and } x - y + 2z = 0 \text{ and } x + 2y - z = 0 \right\}$

N $\left\{ \begin{bmatrix} t^2 \\ t^2 \\ t^2 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

N $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \end{bmatrix} \right\}$

Y $\left\{ \begin{bmatrix} a - 2b \\ a + c \\ 2b - 3c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$

9. [**3 points**] Suppose A is an 8×12 matrix and $\vec{b}$ is a vector in $\mathbb{R}^8$ such that the equation $A\vec{x} = \vec{b}$ is inconsistent. For each statement below, write 'T' if the statement is true, and write 'F' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.5 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

F rank $A = 8$.

T $\text{rank}(A) + \dim \text{Nul}(A) = 12$.

T $\vec{b}$ does not belong to Col A.

F The homogeneous equation $A\vec{x} = \vec{0}$ is inconsistent.

T Nul $A \neq \{\vec{0}\}$.

T At least one row of A does not contain a pivot position.

10. [**2 points**] Write down an eigenvector corresponding to the eigenvalue $\lambda = 1 + i$ for the matrix

$$A = \begin{bmatrix} i+1 & 1 \\ 0 & 2+i \end{bmatrix}.$$

Answer: Any vector of the form $c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for a nonzero scalar $c \in \mathbb{C}$.

11. [**2 points**] Let A and B be $n \times n$ matrices such that A is invertible. Solve the matrix equation $(AX)^T + B - BA^T = 0$ for the matrix X.

Answer: $X = B^T - A^{-1}B^T$

12. [**2 points**] Suppose that $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = -2$. Calculate the following determinant:

$$\begin{vmatrix} 2a_{1,1} & a_{1,2} & a_{1,1} + a_{1,2} + a_{1,3} \\ 2a_{2,1} & a_{2,2} & a_{2,1} + a_{2,2} + a_{2,3} \\ 2a_{3,1} & a_{3,2} & a_{3,1} + a_{3,2} + a_{3,3} \end{vmatrix}.$$

Answer: The determinant is -4.

13. [**2 points**] Suppose

$$A = \begin{bmatrix} 2 & -4 & 5 & 0 & 7 & 6 & -1 \\ 0 & -3 & -8 & 2 & 1 & 6 & -3 \\ 0 & 0 & 0 & 0 & -4 & 1 & 2 \end{bmatrix}$$

and

$$W = \{\vec{x} \in \mathbb{R}^7 \mid A\vec{x} = \vec{0}\}.$$

What is the dimension of W?

Answer: $W = \operatorname{Nul} A$, and so $\dim W = 4$ (the number of non-pivot columns of A).

Part B: Long Answer Questions

For Questions 14–21, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

14.
(a) [**3 pts**] Consider the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}.$$

Find the general solution to the vector equation

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0}.$$

Write your answer in vector parametric form.

Solution: We reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 0 & 5 & 0 \\ -1 & -2 & 3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -2 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{5}{3}R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the general solution is:

$$\begin{aligned} x_1 &= -2x_2 \\ x_2 &\quad \text{free} \\ x_3 &= 0 \end{aligned}$$

The solution in vector parametric form is

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R}.$$

(b) [**1 pt**] Notice that

$$\vec{u}+\vec{v}-\vec{w}=\begin{bmatrix}-5\\-6\\3\end{bmatrix}.$$

Without performing any row reduction, find the general solution to the vector equation

$$x_1\vec{u}+x_2\vec{v}+x_3\vec{w}=\begin{bmatrix}-5\\-6\\3\end{bmatrix}.$$

Solution: We simply add the particular solution $(1,1,-1)$ to the general solution found in part (a). Thus, the general solution to the new equation is

$$\vec{x}=\begin{bmatrix}1\\1\\-1\end{bmatrix}+x_2\begin{bmatrix}-2\\1\\0\end{bmatrix},\quad x_2\in\mathbb{R}.$$

15. [**5 pts**] Calculate the determinant of

$$M = \begin{bmatrix} 3 & 1 & 2 & 2 \\ -2 & 0 & 1 & 1 \\ 5 & 4 & -3 & -3 \\ 10 & -8 & 2 & 1 \end{bmatrix}.$$

Hint: Is there a row or column operation that you can perform to simplify the computation?

Solution: We first subtract the third column from the fourth column, giving

$$\det M = \begin{vmatrix} 3 & 1 & 2 & 0 \\ -2 & 0 & 1 & 0 \\ 5 & 4 & -3 & 0 \\ 10 & -8 & 2 & -1 \end{vmatrix}.$$

Now we expand along the fourth column:

$$\det M = -1 \begin{vmatrix} 3 & 1 & 2 \\ -2 & 0 & 1 \\ 5 & 4 & -3 \end{vmatrix}.$$

Next we expand along the second row:

$$\begin{aligned} \det M &= -1 \left(-(-2) \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \right) \\ &= -1\Big(2(1(-3) - 2 \cdot 4) - 1(3 \cdot 4 - 1 \cdot 5)\Big) \\ &= -1\Big(2(-11) - 1 \cdot 7\Big) = 29 \end{aligned}$$

16. Consider the matrix $B = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) [**3 pts**] Find the eigenvalues of B.

Solution: Starting with an expansion along the first column, we have

$$\det(B - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda).$$

Thus, the eigenvalues are 1 (with multiplicity one) and 3 (with multiplicity 2).

(b) [**4 pts**] For each of the eigenvalues of B found in part (a), find a basis of the corresponding eigenspace.

Solution: For the eigenvalue $\lambda = 1$, we have:

$$\left[\begin{array}{c|c} B - I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For the eigenvalue $\lambda = 3$, we have:

$$\left[\begin{array}{c|c} B - 3I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(c) **[2 pts]** Find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$. You do *not* need to calculate P^{-1}.

Solution: Since the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue, the matrix B is diagonalizable. If

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then we have $B = PDP^{-1}$.

17. In the city of Autoville, there are only two car insurance companies: *Iguana Insurance* and *Camel Coverage*. Initially, each company insures half of the city's drivers. Each year, 20% of Iguana Insurance's customers switch to Camel Coverage and 30% of Camel Coverage's customers switch to Iguana Insurance.

(a) **[1 pt]** Write down the migration matrix M and the initial state vector $\vec{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Another possible migration matrix (if one reverses the order of the companies) is

$$M = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}.$$

(b) **[4 pts]** Find the steady-state vector. In the long term, what fraction of the drivers does each company insure? Remember to justify your answer (i.e. explain *why* the steady state vector describes the long term behaviour).

Solution: To find the steady-state vector, we must find an eigenvector of eigenvalue 1. We row reduce:

$$\left[\begin{array}{c|c} M - I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.2 & 0.3 & 0 \\ 0.2 & -0.3 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -0.2 & 0.3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-5R_1} \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is thus

$$\vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}.$$

We choose the value of the free variable so that the sum of the entries of $\vec{x}$ is equal to one:

$$\left(\frac{3}{2} + 1\right) x_2 = 1 \implies x_2 = \frac{2}{5}.$$

Thus the steady-state vector is

$$\vec{x} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}.$$

Since M is regular stochastic, the long term behaviour is given by the steady-state vector. Thus, in the long term, Iguana Insurance covers 3/5 of the drivers and Camel Coverage insures 2/5 of the drivers.

18. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 4 & 8 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

(a) [**3 points**] Find a basis for Col A.

Solution: We row reduce to find an echelon form of A.

$$A \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & -2 & 6 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{-1/2R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_4} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -2 & -1 \end{bmatrix} \xrightarrow{R_3+R_4} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the pivot columns are columns 1, 3, and 4. Thus, a basis of Col A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) [**2 points**] What are rank A and dim Nul A?

Solution: Since A has three pivot columns, rank $A = 3$. Since A has 2 non-pivot columns, dim Nul $A = 2$.

19. Let

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 4 & 2 \\ 1 & 8 & 3 \end{bmatrix}.$$

(a) [**4 points**] Is A invertible? If so, find its inverse.

Solution: We row reduce the super-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 1 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 9 & -6 & -2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{array}\right] \xrightarrow{-5R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 3 & 1 \end{array}\right]$$

Thus A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} 4 & -1 & -2 \\ 1 & -1 & 0 \\ -4 & 3 & 1 \end{bmatrix}.$$

(b) [**2 points**] Let

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Find a matrix B such that $BA = C$.

Solution: We have $B = CA^{-1}$. Thus,

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 & -2 \\ 1 & -1 & 0 \\ -4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 5 & -4 & -1 \\ 6 & -3 & -2 \end{bmatrix}.$$

20. [**6 points**] Consider the vectors

$$\vec{v}_1 = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}3\\7\\0\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}0\\5\\1\\-1\end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix}0\\2\\2\\-2\end{bmatrix}.$$

Are the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ linearly independent? If not, find a linear dependence relation.

Solution: We form the matrix whose columns are the given vectors and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 2 & 7 & 5 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right] \xrightarrow[R_3+R_4]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-5R_3+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -8 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 24 & 0 \\ 0 & 1 & 0 & -8 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a free variable, the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are not linearly independent (i.e. they are linearly dependent). To find a linear dependence relation, we find a nontrivial solution to the sytem. The general solution is

$$\vec{x} = x_4 \begin{bmatrix}-24\\8\\-2\\1\end{bmatrix}, \qquad x_4 \in \mathbb{R}.$$

We choose any nonzero value for x_4. For instance, if we take $x_4 = 1$, we obtain the linear dependence relation

$$-24\vec{v}_1 + 8\vec{v}_2 - 2\vec{v}_3 + \vec{v}_4 = 0.$$

21. An economy consists of two sectors: agriculture and mining. In order to produce one unit, the agriculture sector consumes 0.2 units from the agriculture sector and 0.6 units from the mining sector. In addition, in order to produce one unit, the mining sector consumes 0.2 units from the agriculture sector and 0.6 units from the mining sector.

(a) [**1 point**] Write the consumption matrix for this economy.

Solution:

$$C = \begin{bmatrix} 0.2 & 0.2 \\ 0.6 & 0.6 \end{bmatrix}$$

(b) [**1 point**] Write the Leontief Input-Output Model production equation.

Solution: The equation is $\vec{x} = C\vec{x} + \vec{d}$, or $(I - C)\vec{x} = \vec{d}$.

(c) [**3 points**] Determine the production levels needed to satisfy a final demand of 8 units from the agriculture sector and 5 units from the mining sector.

Solution: We solve the equation $A\vec{x} = \vec{d}$, where

$$A = I_2 - C = \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}.$$

We have

$$\det A = 0.8 \cdot 0.4 - 0.6 \cdot 0.2 = 0.32 - 0.12 = 0.2 \neq 0.$$

Thus A is invertible and

$$A^{-1} = \frac{1}{0.2} \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

Therefore,

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 44 \end{bmatrix}.$$

So the agriculture sector needs to produce 21 units and the mining sector needs to produce 44 units.