University of Ottawa
Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
Instructor: Alistair Savage

Third Midterm Test Solutions – White Version
21 March 2014

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	Total
Maximum	3	3	2	4	5	4	4	25
Grade								

QUESTION 1. **[3 points]** Which of the following sets are subspaces of $\mathbb{R}^n$ for the given n? You do not need to justify your answer.

(a) $\{(2a, 3b-4a, -2b+1, 5c) \mid a, b, c \in \mathbb{R}\}$, $n = 4$.
(b) $\{(2x, 0, -3y+x) \mid x, y \in \mathbb{R}\}$, $n = 3$.
(c) The column space of a 6×3 matrix, $n = 6$.
(d) The null space of a 5×4 matrix, $n = 5$.
(e) The solution set of a homogeneous system in 4 variables with 6 equations, $n = 4$.
(f) $\{(x, y) \mid 2x + y = 2\}$, $n = 2$.

Solution: (b), (c), (e)

QUESTION 2. **[3 points]** Compute the following and write your answer in the form $a + bi$, $a, b \in \mathbb{R}$.

(a) $\begin{vmatrix} 2+3i & 4i \\ -3 & 3-i \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2+3i & 4i \\ -3 & 3-i \end{vmatrix} = (2+3i)(3-i) - (4i)(-3) = 6 - 2i + 9i + 3 + 12i = 9 + 19i$$

(b) $\dfrac{2-i}{3+4i}$

Solution:

$$\frac{2-i}{3+4i} = \frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{6-8i-3i-4}{9+16} = \frac{2}{25} - \frac{11}{25}i$$

QUESTION 3. [**2 points**] Which of the following statements are true? Note that more than one statement may be true. You should indicate *all* the true statements. (You will lose points for indicating that false statements are true, but you cannot receive a negative score on this question.)

(a) The nullity of a matrix is the number of pivot columns of that matrix.
(b) If A is an $m \times n$ matrix, then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.
(c) An $n \times n$ matrix is invertible if and only if $\operatorname{rank} A = n$.
(d) A square matrix is invertible if and only if its transpose is invertible.
(e) If A is a square matrix, then its column space has the same dimension as its null space.
(f) The number $5 + 4i$ is an imaginary number.

Solution: (b), (c), (d)

Question 4.

(a) [**2 points**] Suppose

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 5.$$

What is

$$\begin{vmatrix} -4a_{11} & a_{13} & a_{12} \\ -4a_{21} & a_{23} & a_{22} \\ -4a_{31} & a_{33} & a_{32} \end{vmatrix} ?$$

Solution: Since the second matrix is obtained from the first by multiplying the first column by -4 and swapping the second and third columns, the determinant of the second matrix is $(-4)(-1)(5) = 20$.

(b) [**2 points**] Suppose A, B, and C are 4×4 matrices with $\det A = -1$, $\det B = 3$, and $-2B^{-1}A^T CB^T = I$. What is the determinant of C?

Solution: We have

$$\begin{aligned} &\det(-2B^{-1}A^T CB^T) = \det I = 1 \\ \Longrightarrow\ &(-2)^4(\det B)^{-1}(\det A^T)(\det C)(\det B^T) = 1 \\ \Longrightarrow\ &16(\det A)(\det C) = 1 \\ \Longrightarrow\ &16(-1)(\det C) = 1 \\ \Longrightarrow\ &\det C = \frac{-1}{16}. \end{aligned}$$

QUESTION 5. [**5 points**] Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 6 & 0 & -1 & 0 \\ 1 & 8 & 0 & 3 & 1 \\ 0 & 0 & 5 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 6 & -3 & -2 & 3 & 0 \end{bmatrix}.$$

Solution: We expand along the fourth row to get

$$\det A = 2 \cdot \begin{vmatrix} 2 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 5 & 0 & 2 \\ 6 & -2 & 3 & 0 \end{vmatrix}.$$

Next, we expand along the second column:

$$\det A = 2 \left((-5) \begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ 6 & 3 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} \right).$$

Finally, we expand the first determinant above along the third column and the second determinant along the third row to get

$$\det A = -10(-1) \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} - 4(2) \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 120 - 56 = 64.$$

QUESTION 6. [**4 points**] Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis of $\operatorname{Nul} A$.

Solution: We row reduce:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3+R_2]{2R_3+R_1} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 \\ 0 & 0 & 2 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 6 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The reduced matrix corresponds to the linear system:

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 6x_2 & & & + & 3x_4 & & & = & 0 \\ & & & + & x_3 & - & 4x_4 & & & = & 0 \\ & & & & & & & & x_5 & = & 0 \end{array}$$

The general solution is

$$\begin{aligned} x_1 &= -6x_2 - 3x_4 \\ x_3 &= 4x_4 \\ x_5 &= 0 \\ x_2, x_4 &\ \text{ free} \end{aligned}$$

The solution in vector parametric form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$$

Thus, a basis for $\operatorname{Nul} A$ is

$$\left\{ \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) What is the dimension of $\operatorname{Nul} A$?

Solution: $\dim \operatorname{Nul} A = 2$

QUESTION 7. [**4 points**] Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & -3 & -6 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

(a) Find a basis for Col A.

Solution: We reduce the matrix to echelon form:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & -3 & -6 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & -3 & -6 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{\frac{2}{3}R_2+R_3} \begin{bmatrix} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & -3 & -6 \\ 0 & 0 & 0 & 0 & -4 & 2 \end{bmatrix}.$$

The first, second, and fifth columns are the pivot columns and so the corresponding columns of A form a basis of Col A:

$$\left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 6 \end{bmatrix} \right\}.$$

(b) What is the rank of A?

Solution: $\operatorname{rank} A = 3$