University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
Instructor: Alistair Savage

Second Midterm Test Solutions – White Version
28 February 2014

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	2	5	6	6	6	29
Grade							

QUESTION 1. [**4 points**] Suppose

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Compute each of the following (or state that the operations are not defined).

(a) $AB - BA$

Solution: $AB - BA$ is not defined.

(b) AC

Solution:

$$AC = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -5 & 3 \end{bmatrix}$$

(c) $2B - C^T$

Solution:

$$2B - C^T = 2\begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -3 \\ 1 & 5 & 3 \end{bmatrix}$$

QUESTION 2. [**2 points**] Which of the following statements are true? Note that more than one statement may be true. You should indicate *all* the true statements. (You will lose points for indicating that false statements are true, but you cannot receive a negative score on this question.)

(a) Any list of linearly dependent vectors contains two parallel vectors.
(b) If A is a square matrix and the matrix-vector equation $A\vec{x} = \vec{b}$ has a solution, then A is invertible.
(c) If A is an $n \times n$ invertible matrix, then the matrix-vector equation $A\vec{x} = \vec{b}$ has a solution for *every* $\vec{b} \in \mathbb{R}^n$.
(d) For all square matrices A and B, it is true that $(AB)^T = A^T B^T$.
(e) Any list of five vectors in $\mathbb{R}^4$ is linearly dependent.
(f) If A, B, and C are matrices such that $AB = AC$, then $B = C$.

Solution: (c), (e)

QUESTION 3.

(a) **[3 points]** Suppose A, B, C and X are square matrices of the same size, and that A, B and C are invertible. Solve the following matrix equation for X.

$$B^{-1}C^{-1}(X+2I)C^2A - A = 0$$

Solution: We have

$$\begin{aligned}
& B^{-1}C^{-1}(X+2I)C^2A - A = 0 \\
\implies & B^{-1}C^{-1}(X+2I)C^2A = A \\
\implies & B^{-1}C^{-1}(X+2I)C^2AA^{-1} = AA^{-1} \\
\implies & B^{-1}C^{-1}(X+2I)C^2 = I \\
\implies & BB^{-1}C^{-1}(X+2I)C^2 = BI \\
\implies & C^{-1}(X+2I)C^2 = B \\
\implies & CC^{-1}(X+2I)C^2 = CB \\
\implies & (X+2I)C^2 = CB \\
\implies & (X+2I)C^2C^{-2} = CBC^{-2} \\
\implies & X+2I = CBC^{-2} \\
\implies & X = CBC^{-2} - 2I
\end{aligned}$$

(b) **[2 points]** Suppose A, B, and C are invertible matrices of the same size. Simplify the following expression as much as possible (i.e. find an equivalent expression involving as few matrices as possible).

$$C^{-1}A(BA)^{-1}BC^TC(AB)^T(A^{-1})^T$$

Solution: We have

$$\begin{aligned}
C^{-1}A(BA)^{-1}BC^TC(AB)^T(A^{-1})^T &= C^{-1}AA^{-1}B^{-1}BC^TCB^TA^T(A^T)^{-1} \\
&= C^{-1}I^2C^TCB^TI \\
&= C^{-1}C^TCB^T
\end{aligned}$$

Question 4.

(a) [**4 points**] Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

invertible? If so, find its inverse.

Solution: We write down the superaugmented matrix and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3+R_2]{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & -1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Therefore, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(b) [**2 points**] Suppose A is the matrix from part (a) and B is a 3×3 matrix such that

$$AB = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Find the matrix B (i.e. write it down explicitly).

Solution: Since A is invertible, we have

$$B = A^{-1}(AB) = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ -3 & -2 & 3 \\ 2 & 1 & -2 \end{bmatrix}.$$

Question 5. Consider a closed economy consisting of two sectors: service and agriculture. In order to produce one unit of output, the service sector needs to consume 0.8 units from its own sector and 0.1 units from agriculture. On the other hand, the agriculture sector needs to consume 0.7 units from its own sector and 0.2 units from the service sector to produce one unit of output.

(a) [**1 point**] Write down the consumption matrix for this economy.

Solution: If you order the sectors by placing service first, then agriculture, the consumption matrix is

$$C = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}.$$

If you order the sectors by placing agriculture first, then service, the consumption matrix is

$$C = \begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}.$$

(For the remainder of these solutions, we will use the first consumption matrix. That is, we order the sectors by placing service first.)

(b) [**1 point**] What are the intermediate demands created if agriculture plans to produce 200 units?

Solution: We multiply the unit consumption vector for agriculture by 200:

$$200 \begin{bmatrix} 0.2 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 40 \\ 140 \end{bmatrix}.$$

Thus, the intermediate demand is 40 units from the service sector and 140 units from agriculture.

(c) [**4 points**] Determine the production levels needed to satisfy a final demand of 80 units from the service sector and 60 units from the agriculture sector.

Solution: We must solve the production equation $(I - C)\vec{x} = \vec{d}$, where

$$\vec{d} = \begin{bmatrix} 80 \\ 60 \end{bmatrix}.$$

We row reduce

$$\left[\, I - C \,\middle|\, \vec{d} \,\right] = \left[\begin{array}{cc|c} 0.2 & -0.2 & 80 \\ -0.1 & 0.3 & 60 \end{array}\right] \xrightarrow[10R_2]{10R_1} \left[\begin{array}{cc|c} 2 & -2 & 800 \\ -1 & 3 & 600 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & -2 & 800 \\ 0 & 2 & 1000 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 2 & -2 & 800 \\ 0 & 1 & 500 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_1} \left[\begin{array}{cc|c} 2 & 0 & 1800 \\ 0 & 1 & 500 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 0 & 900 \\ 0 & 1 & 500 \end{array}\right]$$

Therefore, to meet the given final demands, the service sector must produce 900 units and the agriculture sector must produce 500 units.

ALTERNATE METHOD: It is also possible to solve this problem using the inverse matrix method. We have

$$(I-C)^{-1} = \frac{1}{(0.2)(0.3)-(-0.2)(-0.1)} \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 5 & 10/3 \\ 5/3 & 10/3 \end{bmatrix}.$$

Thus,

$$\vec{x} = (I-C)^{-1}\vec{d} = \begin{bmatrix} 7.5 & 5 \\ 2.5 & 5 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \end{bmatrix} = \begin{bmatrix} 900 \\ 500 \end{bmatrix}.$$

Question 6. [**6 points**] Are the vectors

$$(0,0,1,3), \quad (0,1,2,6), \quad (1,0,0,-2), \quad (2,-1,-2,-10)$$

linearly dependent or independent? If they are linearly dependent, find a linear dependence relation.

Solution: We are interested in solutions to the vector equation

$$x_1(0,0,1,3) + x_2(0,1,2,6) + x_3(1,0,0,-2) + x_4(2,-1,-2,-10) = (0,0,0,0).$$

We form the corresponding augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 3 & 6 & -2 & -10 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 3 & 6 & -2 & -10 & 0 \end{array}\right] \xrightarrow{-3R_1+R_4} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{+2R_3+R_4} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the fourth column does not contain a pivot, the equation has nontrivial solutions. Thus the vectors are *linearly dependent*. The general solution to the equation is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= x_4 \\ x_3 &= -2x_4 \\ x_4 &\text{ free} \end{aligned}$$

To find a linear dependence relation, we take any nontrivial solution. Taking, for instance, $x_4 = 1$, gives

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = -2, \quad x_4 = 1.$$

Thus

$$0(0,0,1,3) + (0,1,2,6) - 2(1,0,0,-2) + (2,-1,-2,-10) = (0,0,0,0)$$

is a linear dependence relation amongst the given vectors.