University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

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First Midterm Test Solutions – White Version

31 January 2014

Surname ____________________ First Name ____________________

Student # ____________________ DGD (1–4) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	2	5	5	4	6	25
Grade							

QUESTION 1. Calculate the following.

(a) **[1 point]** $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} =$

Solution: $\begin{bmatrix} 8 \\ 1 \\ 15 \end{bmatrix}$

(b) **[2 points]** $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} =$

Solution: $1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

QUESTION 2. **[2 points]** Which of the following statements are true? Note that more than one statement may be true. You should indicate *all* the true statements. (You will lose points for indicating that false statements are true, but you cannot receive a negative score on this question.)

(a) A homogeneous systems always has infinitely many solutions.
(b) If the coefficient matrix of a linear system is a 4×5 matrix with 4 pivot columns, then the system is consistent.
(c) A matrix may have more than one reduced echelon form.
(d) If one of the rows of an echelon form of an augmented matrix is of the form $[0\ 0\ 0\ 1\ 0]$, then the associated linear system must be consistent.
(e) The span of two nonzero parallel vectors in $\mathbb{R}^3$ is a line.

Solution: (b), (e)

QUESTION 3. [**5 points**] Find all the solutions to the following linear system.

$$\begin{array}{rcrcrcr} & & x_2 & + & 2x_3 & = & 3 \\ x_1 & + & & + & 3x_3 & = & 1 \\ 4x_1 & - & 3x_2 & + & 8x_3 & = & 1 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & -1 \end{array}$$

Solution: We row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 1 \\ 4 & -3 & 8 & 1 \\ 2 & -3 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 4 & -3 & 8 & 1 \\ 2 & -3 & 2 & -1 \end{array}\right] \xrightarrow[-2R_1+R_4]{-4R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -4 & -3 \\ 0 & -3 & -4 & -3 \end{array}\right]$$

$$\xrightarrow{-R_3+R_4} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -4 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_3+R_2]{-3R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, there is a unique solution, given by:

$$x_1 = -8, \quad x_2 = -3, \quad x_3 = 3.$$

QUESTION 4. [**5 pts**] For which values of h and k does the matrix-vector equation

$$\begin{bmatrix} 1 & 2h \\ 3 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ k \end{bmatrix}$$

(a) have no solutions,
(b) have a unique solution,
(c) have infinitely many solutions?

Solution: We reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2h & 2 \\ 3 & 6 & k \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & 2h & 2 \\ 0 & 6-6h & k-6 \end{array}\right]$$

(a) The system has no solution when $h = 1$ and $k \neq 6$.
(b) The system has a unique solution when $h \neq 1$ and $k \in \mathbb{R}$.
(c) The system has infinitely many solutions when $h = 1$ and $k = 6$.

QUESTION 5. [**4 pts**] Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -1 \\ 4 \\ 17 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} c \\ 0 \\ d \end{bmatrix}.$$

For which values of c and d is the vector $\vec{b}$ a linear combination of the vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$?

Solution: We reduce the augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & c \\ 0 & 1 & 4 & 0 \\ 3 & -1 & 17 & d \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & c \\ 0 & 1 & 4 & 0 \\ 0 & 5 & 20 & d-3c \end{array}\right] \xrightarrow{-5R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & c \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & d-3c \end{array}\right]$$

The equation has a solution when there is no pivot in the last column. Thus, it has a solution if and only if $d - 3c = 0$, that is, if $d = 3c$.

QUESTION 6. [**6 pts**]

(a) Find the general solution to the following linear system:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 0 \\ -x_1 & + & x_2 & & & = & 0 \end{array}$$

Write your answer in *vector parametric form*. What is the geometric interpretation of the solution set?

Solution: We reduce the augmented matrix to reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{-1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The basic variables are x_1 and x_2, while x_3 is a free variable. The linear system corresponding to the reduced matrix is

$$\begin{array}{rcrcrcl} x_1 & & & - & x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 0 \end{array}$$

We thus have:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

This is a line through the origin in the direction $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) Check that

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 0$$

is a solution to the linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 5 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 4 \\ -x_1 & + & x_2 & & & = & 1 \end{array}$$

Without performing any row reduction, find the general solution to this linear system in *vector parametric form*. (Note that the coefficient matrix here is the same as in part (a).)

Solution: We check by direct substitution that $(1, 2, 0)$ is a solution to the system:

$$\begin{array}{rcrcrcr} 1 & + & 2(2) & - & 3(0) & = & 5 \\ 2(1) & + & 2 & - & 3(0) & = & 4 \\ -(1) & + & 2 & & & = & 1 \end{array}$$

Then, we need only add this particular solution to the solutions to the homogeneous system from part (a). Therefore, the general solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$