University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

Instructor: Alistair Savage

Final Exam – Solutions

April 2014

Surname ________________________ First Name ________________________

Student # ________________________ Seat # ________________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 14 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 15 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(g) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.

(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–14	15	16	17	18	19	20	21	22	Total
Maximum	30	5	4	5	9	4	4	4	5	70
Grade										

Part A: Answer Only Questions

For Questions 1–14, only your final answer will be considered for marks. Write your final answers in the spaces provided.

1. **[2 pts]** Find a vector equation of the line passing through the points $(3, -2, 0)$ and $(1, -4, 5)$.

Answer: A vector equation of the line is $(3, -2, 0) + t(-2, -2, 5)$, $t \in \mathbb{R}$. (Other answers are possible.)

2. **[2 pts]** If

$$A = \begin{bmatrix} 0 & 2 & -3 \\ 2 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 3 \end{bmatrix},$$

what are AB and $A^T - 2B$?

Answer: $AB = \begin{bmatrix} 4 & -13 \\ 4 & 12 \end{bmatrix}$ and $A^T - 2B = \begin{bmatrix} -2 & 0 \\ -2 & 5 \\ -3 & -2 \end{bmatrix}$.

3. **[2 pts]** If $w = 2 - i$ and $z = 4 + 2i$, what are $w\bar{z}$ and $w + 2z$? Write your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Answer: $w\bar{z} = 6 - 8i$, and $w + 2z = 10 + 3i$.

4. **[2 pts]** Compute

$$\frac{3+i}{3-2i}.$$

Write your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

Answer: $\frac{7}{13} + \frac{9}{13}i$

5. **[2 pts]** Suppose that P, Q and R are invertible matrices. Solve the equation

$$R^T(X + Q^{-1})PQ - Q = 0$$

for the matrix X, i.e. express X in terms of P, Q, R, their inverses and their transposes. You may assume that the sizes of the matrices are such that all of the operations are defined.

Answer: $X = (R^T)^{-1}P^{-1} - Q^{-1}$.

6. [**2 pts**] Each of the items below describes a hypothetical linear system. For each one, if it is not possible for such a linear system to exist, write 'DNE'. If such a linear system is possible, but would be inconsistent, write 'INC'. Otherwise, write the number of parameters that would be needed to describe the solution set of the system.

1 A linear system consisting of 5 equations in 4 unknowns whose coefficient matrix and augmented matrix both have rank 3.

DNE A linear system consisting of 7 equations in 3 unknowns, whose coefficient matrix and augmented matrix both have rank 4.

INC A linear system consisting of 6 equations in 8 unknowns, whose coefficient matrix has rank 5 and whose augmented matrix has rank 6.

2 A linear system consisting of 5 equations in 5 unknowns, whose coefficient and augmented matrix both have rank 3.

7. [**3 pts**] Suppose A is an $n \times n$ matrix with real number entries, and consider the statement:

$$\text{The matrix } A \text{ is invertible. } (*)$$

For each of the assertions below, write 'Y' if it is equivalent to the statement $(*)$ and 'N' if it is not.

Y The matrix A has n pivot positions.

Y The columns of A form a basis of $\mathbb{R}^n$.

N The matrix A is diagonalizable.

Y The nullity of A is zero.

N The determinant of A is equal to one.

Y The number zero is not an eigenvalue of A.

8. [**3 pts**] For each statement below, indicate if it is true (T) or false (F).

T A list of vectors is linearly dependent if and only if one of them is a linear combination of the others.

F An $n \times n$ matrix A is invertible if and only if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

F A matrix may have the zero vector as an eigenvector.

T Every subspace of $\mathbb{R}^n$ can be written as a span of some vectors in $\mathbb{R}^n$.

F If A is a square matrix, then the eigenvectors of A and of A^T are the same.

F If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

9. [**3 pts**] For each statement below, indicate if it is true (T) or false (F).

F It is possible for a linear system to have exactly 5 solutions.

T A matrix is invertible if and only if zero is not a root of its characteristic polynomial.

F Every matrix has exactly one row echelon form.

F Every nonzero matrix has an inverse.

T The determinant of a triangular matrix is the product of the entries on its diagonal.

F If A and B are $n \times n$ matrices such that AB is equal to the $n \times n$ zero matrix, then either A or B is the zero matrix.

10. [**3 pts**] For each of the following sets, write 'Y' if it is a subspace of $\mathbb{R}^n$ for the given n, and write 'N' if it is not.

Y An eigenspace (corresponding to some eigenvalue) of an $n \times n$ matrix A.

Y $\{(2u - 3v, u + 5v, u) \mid u, v \in \mathbb{R}\}$, $n = 3$.

N $\{(a^2, a) \mid a \in \mathbb{R}\}$, $n = 2$.

N $\{(b, -b) \mid b \in \mathbb{Z}\}$, $n = 2$. (Recall that $\mathbb{Z}$ is the set of integers.)

N $\{\vec{x} \mid A\vec{x} = \vec{0}\}$, where A is a 5×8 matrix, $n = 5$.

Y $\{(x - 1, x - 1) \mid x \in \mathbb{R}\}$, $n = 2$.

11. [**1 pt**] If A is an $n \times n$ matrix and c is a real number, write an expression for $\det(cA)$ in terms of c and $\det A$.

Answer: $\det(cA) = c^n \det A$.

12. [**2 pts**] What are the eigenvalues of the following matrix? For each eigenvalue, give its multiplicity.

$$\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 3+i & -i & 0 & 0 & 0 \\ 5-i & 4 & 0 & 0 & 0 \\ 2 & 5i & -3 & i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer: The eigenvalues are 0 (multiplicity one), 1 (multiplicity one), $-i$ (multiplicity one) and i (multiplicity two).

13. [**1 pt**] Write down the Leontief Input-Output Model Production Equation.

Answer: $\vec{x} = C\vec{x} + \vec{d}$ or $(I - C)\vec{x} = \vec{d}$.

14. [**2 pts**] Suppose an economy consists of three sectors: A, B, and C. In order to produce one unit of output, sector A must consume 0.4 units from itself and 0.2 units from each of sectors B and C. In order to produce one unit of output, sector B must consume 0.3 units from sector A, 0.2 units from itself, and 0.1 units from sector C. In order to produce one unit of output, sector C must consume 0.4 units from sector A, no units from sector B, and 0.25 units from itself. Write down the consumption matrix for this economy.

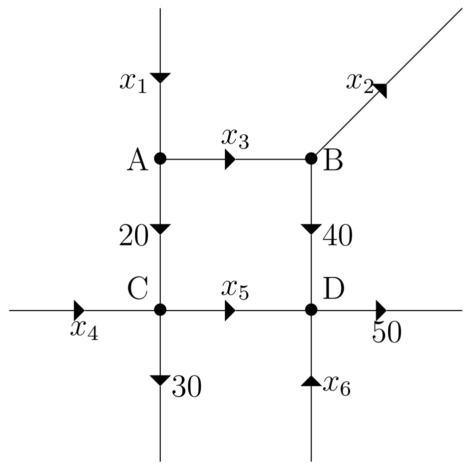
Answer: $\begin{bmatrix} 0.4 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0 \\ 0.2 & 0.1 & 0.25 \end{bmatrix}$

Part B: Long Answer Questions

For Questions 15–22, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

15. **[5 pts]**

(a) Give a system of equations describing the flow in the following network. The arrows indicate the direction of flow. The letters A through D label intersections. Include all relevant equations. Do *not* solve the system.



Solution: Setting the total flow in equal to the flow out at each intersection and for the overall system as a whole, we get the following linear system.

Intersection	**Flow in**		**Flow out**
A	x_1	$=$	$x_3 + 20$
B	x_3	$=$	$x_2 + 40$
C	$x_4 + 20$	$=$	$x_5 + 30$
D	$x_5 + x_6 + 40$	$=$	50
Overall	$x_1 + x_4 + x_6$	$=$	$x_2 + 30 + 50$

(b) The reduced row echelon form of the augmented matrix for the linear system from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 0 & 0 & -40 \\ 0 & 0 & 0 & 1 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Write the general solution for the flow in the network. Indicate the ranges of possible values of any free variables (that is, give any minimum and maximum values of any free variables).

Solution: The general solution is:

$$\begin{aligned} x_1 &= x_3 + 20 \\ x_2 &= x_3 - 40 \\ x_3 &\text{ free} \\ x_4 &= 20 - x_6 \\ x_5 &= 10 - x_6 \\ x_6 &\text{ free} \end{aligned}$$

Since the values of all the variables must be nonnegative, the possible values of the free variables are

$$x_3 \geq 40 \quad \text{and} \quad 0 \leq x_6 \leq 10.$$

16.

(a) **[3 pts]** Suppose

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ 0 & 2 & -2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 7 \\ 8 \\ 6 \end{bmatrix}.$$

Solve the matrix equation $A\vec{x} = \vec{b}$ and write your solution in vector parametric form.

Solution: We reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 7 \\ 1 & 2 & -3 & 8 \\ 0 & 2 & -2 & 6 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 2 & 1 & -3 & 7 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{-2R_2+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & -3 & 3 & 9 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the general solution is:

$$\begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= 3 + x_3 \\ x_3 &\quad \text{free} \end{aligned}$$

The solution set in vector parametric form is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

(b) **[1 pt]** *Without performing any row reduction*, write the general solution to the matrix equation $A\vec{x} = \vec{0}$ in vector parametric form. Here, A is the same matrix as in part (a).

Solution: We simply remove the particular solution $(2, 3, 0)$ from the general solution to part (a). Thus, the general solution to $A\vec{x} = \vec{0}$ is

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

17. [**5 pts**] Calculate the determinant of

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 & 0 \\ 8 & 0 & -1 & 4 & 0 \\ 7 & -6 & 5 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 4 & 2 & 2 & 0 \end{bmatrix}.$$

Solution: We first expand along the fifth column:

$$\det A = 1 \begin{vmatrix} 1 & 3 & 2 & -1 \\ 8 & 0 & -1 & 4 \\ 3 & 0 & 0 & 0 \\ 3 & 4 & 2 & 2 \end{vmatrix}.$$

Next, we expand along the third row:

$$\det A = 3 \begin{vmatrix} 3 & 2 & -1 \\ 0 & -1 & 4 \\ 4 & 2 & 2 \end{vmatrix}.$$

Now we expand along the first column to get

$$\begin{aligned} \det A &= 3\left(3 \begin{vmatrix} -1 & 4 \\ 2 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\right) \\ &= 3(-30 + 28) = -6. \end{aligned}$$

18. Consider the matrix $A = \begin{bmatrix} -1 & -6 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$.

(a) [**3 pts**] Find the eigenvalues of A.

Solution: Expanding first along the third column, we have

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -6 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & -6 \\ 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)^2.$$

Thus, the eigenvalues are -1 (with multiplicity one) and 2 (with multiplicity 2).

(b) [**4 pts**] For each of the eigenvalues of A found in part (a), find a basis of the corresponding eigenspace.

Solution: For the eigenvalue $\lambda = -1$, we have:

$$\left[\begin{array}{ccc|c} 0 & -6 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For the eigenvalue $\lambda = 2$, we have:

$$\left[\begin{array}{ccc|c} -3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 + R_3} \left[\begin{array}{ccc|c} -3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R},$$

and a basis of this eigenspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) **[2 pts]** Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. You do *not* need to calculate P^{-1}.

Solution: Since the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue, the matrix A is diagonalizable. If

$$P = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

then we have $A = PDP^{-1}$.

19. [**4 pts**] Is the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 3 & -4 & 7 & 0 \\ 5 & -2 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

invertible? If so, find its inverse A^{-1}.

Solution: We row reduce the superaugmented matrix:

$$\left[\ A \mid I_4\ \right] = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 3 & -4 & 7 & 0 & 0 & 1 & 0 & 0 \\ 5 & -2 & 6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -6 & -3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -9 & -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3R_2+R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -6 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -27 & -14 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{6R_4+R_2 \\ -2R_4+R_1}]{27R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & -2 \\ 0 & -1 & 1 & 0 & -3 & 1 & 0 & 6 \\ 0 & 0 & -1 & 0 & -14 & 3 & 1 & 27 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1]{R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -27 & 6 & 2 & 52 \\ 0 & -1 & 0 & 0 & -17 & 4 & 1 & 33 \\ 0 & 0 & -1 & 0 & -14 & 3 & 1 & 27 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-R_2 \\ -R_3}]{-R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -10 & 2 & 1 & 19 \\ 0 & 1 & 0 & 0 & 17 & -4 & -1 & -33 \\ 0 & 0 & 1 & 0 & 14 & -3 & -1 & -27 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Thus, A is invertible and

$$A^{-1} = \begin{bmatrix} -10 & 2 & 1 & 19 \\ 17 & -4 & -1 & -33 \\ 14 & -3 & -1 & -27 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

20. [**4 pts**] Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

linearly dependent? If yes, find a linear dependence relation.

Solution: We reduce the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{-R_1+R_2 \\ -2R_1+R_3 \\ -3R_1+R_4}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{-R_2+R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_3+R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column does not contain a pivot, the corresponding homogeneous linear system has nontrivial solutions. Thus, the vectors are linearly dependent. To find a linear dependence relation, we need to find one nontrivial solution. The general solution is

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= -2x_4 \\ x_3 &= -x_4 \end{aligned}$$

Taking $x_4 = 1$ (we can choose any nonzero value we like), we have $x_1 = 1$, $x_2 = -2$, and $x_3 = -1$. These are the coefficients in a linear dependence relation:

$$\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 + \vec{v}_4 = \vec{0}.$$

21. [**4 pts**] Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 5 \\ -1 \\ 11 \\ -13 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} 4 \\ -1 \\ 9 \\ -10 \end{bmatrix}.$$

What is the dimension of this subspace?

Solution: We reduce the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, and $\vec{v}_5$.

$$\begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 1 & 1 & -1 & -1 \\ 2 & -4 & -2 & 11 & 9 \\ -3 & 2 & -1 & -13 & -10 \end{bmatrix} \xrightarrow[3R_1+R_4]{-2R_1+R_3} \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -4 & -4 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{4R_2+R_4} \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{bmatrix} \xrightarrow{2R_3+R_4} \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns are pivot columns. Thus, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_4$ form a basis of the subspace and its dimension is three.

22. Humans have recently established a colony on Mars. Currently, there are no humans on Mars. Each year, 5% of Earth's population is adventurous and moves to Mars. On the other hand, each year, 60% of Mars' population gets homesick and returns to Earth. For the purposes of this question, ignore changes in the population due to births and deaths.

(a) **[1 pt]** Give the migration matrix M and the initial state vector $\vec{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} .95 & .6 \\ .05 & .4 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) **[4 pts]** Find the steady-state vector. What fraction of the human population lives on Mars in the long term? Remember to justify your answer (i.e. explain *why* the steady state vector describes the long term behaviour).

Solution: To find the steady-state vector, we must find an eigenvector of eigenvalue 1. We row reduce:

$$\left[\begin{array}{c|c} M-I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -.05 & .6 & 0 \\ .05 & -.6 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -.05 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-20R_1} \left[\begin{array}{cc|c} 1 & -12 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is thus

$$\vec{x} = x_2 \begin{bmatrix} 12 \\ 1 \end{bmatrix}.$$

We choose the value of the free variable so that the sum of the entries of $\vec{x}$ is equal to one:

$$(12+1)x_2 = 1 \implies x_2 = 1/13.$$

Thus the steady-state vector is

$$\vec{x} = \begin{bmatrix} 12/13 \\ 1/13 \end{bmatrix}.$$

Since M is regular stochastic, the long term behaviour is given by the steady-state vector. Thus, in the long term, 1/13 of the human population lives on Mars (until the people of Jupiter invade).