University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

Instructor: Alistair Savage

Third Midterm Test Solutions – White Version

26 March 2013

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	Total
Maximum	2	2	3	2	2	6	5	6	28
Grade									

QUESTION 1. [**2 points**] Which of the following subsets are subspaces of $\mathbb{R}^n$? Note that more than one set may be a subspace. You should indicate *all* the subspaces. (You will lose points for incorrect answers, but you cannot receive a negative score on this question.)

(a) The set of solutions to the matrix-vector equation $A\vec{x} = \vec{b}$, where A is an $m \times n$ matrix and $\vec{b}$ is a nonzero vector in $\mathbb{R}^m$.
(b) The set of solutions to the matrix-vector equation $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix.
(c) The set $\{(2a - 3b, b + 7a, 8a + b) \mid a, b \in \mathbb{R}\}$. Here $n = 3$.
(d) The set $\{(c + 5, c) \mid c \in \mathbb{R}\}$. Here $n = 2$.
(e) The span of 5 vectors in $\mathbb{R}^4$. Here $n = 4$.
(f) The set $\{(x^2, x) \mid x \in \mathbb{R}\}$. Here $n = 2$.

Answer: (b), (c), (e)

QUESTION 2. [**2 points**] Which of the following statements are false? Note that more than one statement may be false. You should indicate *all* the false statements. (You will lose points for indicating that true statements are false, but you cannot receive a negative score on this question.)

(a) A square matrix A is invertible if and only if $\det A \neq 0$.
(b) If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors of a matrix A that correspond to distinct eigenvalues, then these vectors are linearly independent.
(c) The eigenvalues of any square matrix are the entries on its diagonal.
(d) For every matrix A, we have $\operatorname{rank} A = \dim \operatorname{Nul} A$.
(e) For every square matrix A, we have $\det A = \det A^T$.
(f) For any three vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ in $\mathbb{R}^n$, the subspace $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ has dimension 3.

Answer: (c), (d), (f)

QUESTION 3. Consider the matrix

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 8 & 3 & 0 \\ 8 & 11 & -10 & 0 \end{bmatrix}.$$

(a) [**2 points**] What are the eigenvalues of M^2?

Answer: 0, 1, 4 and 9

(b) [**1 point**] What is $\det M^2$?

Answer: $\det M^2 = 0$

QUESTION 4. [**2 points**] Suppose A and B are 3×3 matrices with $\det A = 2$ and $\det B = -1$. Calculate $\det(-A^2 B^T A)$.

Solution: We have
$\det(-A^2B^TA) = (-1)^3(\det(A^2))(\det(B^T))(\det A) = -(\det A)^2(\det B)(\det A) = -2^2(-1)2 = 8.$

QUESTION 5. [**2 points**] Consider the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Suppose that $\det A = 4$. What is $\det B$, where

$$B = \begin{bmatrix} a & b & c \\ d-5a & e-5b & f-5c \\ 3g & 3h & 3i \end{bmatrix}?$$

Solution: Since B is obtained from A by multiplying the third row by 3 and adding -5 times the first row to the second row, we have $\det B = 3 \cdot 4 = 12$.

QUESTION 6. [**6 points**] The eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

are 1 and 5. Find bases for the corresponding eigenspaces.

Solution: For the eigenvalue 1, we must find a basis for the null space of $A - I$. So we row reduce

$$\left[\begin{array}{c|c} A - I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is:

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ x_2, x_3 &\text{ free} \end{aligned}$$

In vector parametric form, the solution set is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = -x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus a basis for the eigenspace corresponding to the eigenvalue 1 is

$$\{(-1, 1, 0), (-1, 0, 1)\}.$$

For the eigenvalue 5, we must find a basis for the null space of $A - 5I$. So we row reduce

$$\left[\begin{array}{c|c} A - 5I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right] \xrightarrow[\frac{1}{3}R_1+R_3]{\frac{2}{3}R_1+R_2} \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{8}{3} & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{3}{4}R_2} \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is:

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 2x_3 \\ x_3 &\text{ free} \end{aligned}$$

In vector parametric form, the solution set is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus a basis for the eigenspace corresponding to the eigenvalue 5 is

$$\{(1, 2, 1)\}.$$

QUESTION 7. [**5 points**] Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 8 & 3 \\ 2 & 0 & 0 & -7 & -1 \\ -4 & 6 & 2 & 5 & 3 \\ 3 & -2 & 0 & 13 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution: We first expand along the fifth row (one could also expand along the third column) to get

$$\det A = -1 \cdot \begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 0 & 0 & -1 \\ -4 & 6 & 2 & 3 \\ 3 & -2 & 0 & 1 \end{vmatrix}.$$

Next we expand along the third column to get

$$\det A = -2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -2 & 1 \end{vmatrix}.$$

Then we expand along the second row (one could also expand along the second column) to get

$$\begin{aligned} \det A &= -2 \left(-2 \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \right) \\ &= 4 \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \\ &= 4(2 \cdot 1 - (-2)3) - 2(1(-2) - 3 \cdot 2) \\ &= 4(8) - 2(-8) = 6 \cdot 8 = 48. \end{aligned}$$

QUESTION 8. Consider the matrix

$$A = \begin{bmatrix} -8 & 0 & -24 & -8 & -9 \\ 2 & 1 & 8 & 4 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [**2 points**] Find a basis of Nul A.

Solution: From the reduced echelon form, we see that the general solution is:

$$\begin{aligned} x_1 &= -3x_3 - x_4 \\ x_2 &= -2x_3 - 2x_4 \\ x_3 &\text{ free} \\ x_4 &\text{ free} \\ x_5 &= 0 \end{aligned}$$

Thus the solution set of the homogeneous system in vector parametric form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis of Nul A is

$$\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(b) [**1 point**] What is the dimension of Nul A?

Solution: Since the basis above has two elements, $\dim \operatorname{Nul} A = 2$.

(c) [**2 points**] Find a basis of Col A.

Solution: Since the first, second and last columns of A are its pivot columns, these columns form a basis of $\operatorname{Col} A$:

$$\left\{ \begin{bmatrix} -8 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(d) [**1 point**] What is the rank of A?

Solution: $\operatorname{rank} A = 3$