University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

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First Midterm Test Solutions – White Version

1 February 2013

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	2	3	6	5	4	5	25
Grade							

QUESTION 1. **[2 points]** Which of the following statements are true? Note that more than one statement may be true. You should indicate *all* the true statements. (You will lose points for indicating that false statements are true, but you cannot receive a negative score on this question.)

(a) A homogeneous linear system is always consistent.
(b) Every matrix is row equivalent to exactly one matrix in row echelon form.
(c) Every consistent linear system with 6 equations and 6 variables has exactly one solution.
(d) The span of *any* two vectors in $\mathbb{R}^3$ is a plane.
(e) A linear system is consistent if and only if its coefficient matrix and augmented matrix have the same number of pivot positions.

Answer: (a), (e)

QUESTION 2. Compute the following.

(a) **[1 point]** $\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} =$

Solution: $\begin{bmatrix} 3 \\ -18 \\ -6 \end{bmatrix}$

(b) **[2 points]** $\begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} =$

Solution: $3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ 5 \end{bmatrix}$

Question 3. [**6 pts**] Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ -3 & -7 & 1 \end{bmatrix}.$$

(a) Find all solutions to the homogeneous system $A\vec{x} = \vec{0}$ in *vector parametric form*. What is the geometric interpretation of the solution set?

Solution: We reduce the augmented matrix to reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 1 & 4 & 3 & 0 \\ -3 & -7 & 1 & 0 \end{array}\right] \xrightarrow[3R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The basic variables are x_1 and x_2 and the free variable is x_3. Switching back to equation form, we have:

$$\begin{array}{ccccc} x_1 & & - & 5x_3 & = & 0 \\ & x_2 & + & 2x_3 & = & 0 \\ & & & x_3 & \text{free} & \end{array}$$

We thus have:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

In parametric form:

$$\vec{x} = s \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

The solution set is a line. More precisely, it is the line through the origin parallel to the vector $(5, -2, 1)$.

(b) Suppose that $\vec{b} \in \mathbb{R}^3$ and that

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

is a solution to the equation $A\vec{x} = \vec{b}$. Find all solutions to the equation $A\vec{x} = \vec{b}$ in *vector parametric form*. (Note that the matrix A here is the same matrix as in part (a).)

Solution: We need only add the particular solution to the elements of solution set for the corresponding homogeneous equation, found in part (a). Thus the solution set is

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + s \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

Question 4. [**5 pts**] Find all solutions to the following linear system:

$$\begin{array}{rcrcrcr} & & -x_2 & + & 3x_3 & = & 1 \\ x_1 & + & 4x_2 & + & 2x_3 & = & 10 \\ -x_1 & - & 5x_2 & + & x_3 & = & -9 \\ x_1 & + & 4x_2 & + & 3x_3 & = & 11 \end{array}$$

Solution: We write the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 0 & -1 & 3 & 1 \\ 1 & 4 & 2 & 10 \\ -1 & -5 & 1 & -9 \\ 1 & 4 & 3 & 11 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & 3 & 1 \\ -1 & -5 & 1 & -9 \\ 1 & 4 & 3 & 11 \end{array}\right] \xrightarrow[-R_1+R_4]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-2R_3+R_1]{-3R_3+R_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 8 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-4R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the unique solution to the linear system is

$$x_1 = 0, \quad x_2 = 2, \quad x_3 = 1.$$

QUESTION 5. [**4 pts**] Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 2 & 8 \\ 3 & 11 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

For which values of b_1, b_2 and b_3 does the matrix-vector equation $A\vec{x} = \vec{b}$ have at least one solution? Justify your answer.

Solution: We row reduce the augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ -4 & 2 & 8 & b_2 \\ 3 & 11 & 4 & b_3 \end{array}\right] \xrightarrow[-3R_1+R_3]{4R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -10 & -8 & 4b_1 + b_2 \\ 0 & 20 & 16 & -3b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 4b_1 + b_2 \\ 0 & 0 & 0 & 5b_1 + 2b_2 + b_3 \end{array}\right]$$

The equation has a solution when there is no pivot in the last column. Thus, the equation has a solution if and only if

$$5b_1 + 2b_2 + b_3 = 0.$$

QUESTION 6. [**5 pts**] For which values of a and b does the linear system

$$\begin{array}{rcrcl} 2x_1 & + & 4x_2 & = & b \\ x_1 & + & ax_2 & = & 3 \end{array}$$

(a) have no solutions,
(b) have a unique solution,
(c) have infinitely many solutions?

Solution: We write the augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} 2 & 4 & b \\ 1 & a & 3 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1+R_2} \left[\begin{array}{cc|c} 2 & 4 & b \\ 0 & a-2 & 3-\frac{1}{2}b \end{array}\right]$$

(a) The system has no solution when $a = 2$ and $b \neq 6$.
(b) The system has a unique solution when $a \neq 2$.
(c) The system has infinitely many solutions when $a = 2$ and $b = 6$.