University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II

Instructor: Alistair Savage

Final Exam

April 2013

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 14 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 15 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) The number of points available for each question is indicated in square brackets.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–14	15	16	17	18	19	20	21	22	Total
Maximum	24	5	9	4	4	4	4	5	7	66
Grade										

Part A: Answer Only Questions

For Questions 1–14, only your final answer will be considered for marks. Write your final answer(s) in the space(s) provided.

1. **[1 pt]** If

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 15 & -1 & 0 & 0 & 0 & 0 \\ -8 & 7 & 1/4 & 0 & 0 & 0 \\ \pi & 8 & 4 & 2 & 0 & 0 \\ -15 & 23 & 0 & 7 & 2 & 0 \\ 0 & 21 & 4/5 & -13 & 6 & 5 \end{bmatrix},$$

what is $\det A$?

Answer: $\det A = -15$.

2. **[2 pts]** Find a vector equation of the line passing through the points $(-2, 1, 3)$ and $(5, 1, 7)$.

Answer: A vector equation of the line is $(-2, 1, 3) + t(7, 0, 4)$, $t \in \mathbb{R}$. (Other answers are possible.)

3. **[2 pts]** Give a basis for the null space of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. **[2 pts]** If

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

what are AB^T and $A + 2B$?

Answer: $AB^T = \begin{bmatrix} 3 & -1 \\ 3 & 4 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 2 & 5 & -3 \end{bmatrix}$.

5. **[1 pt]** Suppose M is a 10×8 matrix of rank 3. What is the dimension of $\operatorname{Nul} M$?

Answer: $\dim \operatorname{Nul} M = 5$.

6. **[2 pts]** If the complex number

$$\frac{3-i}{2+4i}$$

is written in the form $a+bi$, with $a, b \in \mathbb{R}$, what are a and b?

Answer: $a = \frac{1}{10}$, $b = \frac{-7}{10}$

7. **[2 pts]** Suppose $z = 3+i$ and $w = 5-2i$. Calculate $z - \bar{w}$ and zw.

Answer: $z - \bar{w} = -2 - i$, and $zw = 17 - i$.

8. **[2 pts]** What are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}?$$

Answer: $\pm i$.

9. **[2 pts]** Suppose that A, B and C are invertible matrices. Solve the following equation for the matrix X, i.e. express X in terms of A, B, C, their inverses and their transposes. You may assume that the size of the matrices are such that all of the operations are defined.

$$CB^T(X+A)C^{-1} = A$$

Answer: $X = (B^T)^{-1}C^{-1}AC - A$.

10. **[2 pts]** Which of the following statements are true for **every** invertible $n \times n$ matrix A? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) The matrix A is row equivalent to the identity matrix.
(b) The equation $A\vec{x} = \vec{0}$ has a nontrivial solution.
(c) The columns of A are linearly dependent.
(d) The rank of A is n.
(e) The determinant of A is nonzero.
(f) The dimension of the null space of A is nonzero.

Answer: (a), (d), (e)

11. **[2 pts]** Which of the following statements are true? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) A given matrix can have more than one row echelon form.
(b) A given matrix can have more than one reduced row echelon form.
(c) If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent, then one of the vectors is a multiple of the other.
(d) Every nonzero square matrix has an inverse.
(e) Every vector space has exactly one basis.

(f) If the matrix B is obtained from the matrix A by adding a multiple of one row to another, then $\det A = \det B$.

Answer: (a), (f)

12. **[2 pts]** Which of the following subsets are subspaces of $\mathbb{R}^n$ for the given n? Note that more than one subset may be a subspace. You should indicate **all** the subspaces. (You will lose points for including sets that are not subspaces.)

(a) The solution set of a system of homogenous equations in n variables.
(b) The set of eigenvectors of an $n \times n$ matrix.
(c) $\{(x_1, x_2, x_3) \mid 2x_1 + x_2 = x_3 \text{ and } x_2 = x_3 + 1\}$, $n = 3$.
(d) $\{(y^3, y) \mid y \in \mathbb{R}\}$, $n = 2$.
(e) The set of all linear combinations of three given vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^4$, $n = 4$.

Answer: (a), (e)

13. **[1 pt]** Suppose A and B are 2×2 matrices such that $\det A = 3$ and $\det B = -1$. What is $\det(2A^T BA^{-1})$?

Answer: $\det(2A^T BA^{-1}) = -4$.

14. **[1 pt]** For which values of t (if any) are the vectors

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4 \\ t \end{bmatrix}$$

linearly dependent?

Answer: $t = -6$

Part B: Long Answer Questions

For Questions 15–22, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

15. [**5 points**] Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 3 & 0 & 1 & 0 & -2 \\ -2 & 3 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 10 & 0 \end{bmatrix}.$$

Solution: We first expand along the fourth row:

$$\det A = 1 \begin{vmatrix} 1 & -1 & 2 & 3 \\ 3 & 1 & 0 & -2 \\ -2 & 1 & 2 & 4 \\ 0 & 0 & 10 & 0 \end{vmatrix}.$$

We again expand along the fourth row:

$$\det A = -10 \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & -2 \\ -2 & 1 & 4 \end{vmatrix}.$$

Next we expand along the first row:

$$\begin{aligned} \det A &= -10 \left(1 \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -2 \\ -2 & 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \right) \\ &= -10(1 \cdot 4 - (-2)1) + (3 \cdot 4 - (-2)(-2)) + 3(3 \cdot 1 - (-2)1)) \\ &= -10(6 + 8 + 15) = -290. \end{aligned}$$

16. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) [**3 points**] Find the eigenvalues of A.

Solution: Expanding along the third column, we have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2.$$

Thus the eigenvalues of A are 2 and 3.

(b) [**4 points**] For each eigenvalue, find a basis for the corresponding eigenspace. (*Note:* There is more space for your answer to this part on the following page.)

Solution: For the eigenvalue 2, we row reduce

$$\left[\begin{array}{c|c} A - 2I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace consists of the vectors

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 \text{ free.}$$

Therefore, a basis of this eigenspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For the eigenvalue 3, we row reduce

$$\left[\begin{array}{c|c} A - 3I & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the eigenspace consists of the vectors

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \text{ free.}$$

Therefore, a basis of this eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) [**2 points**] Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(You do *not* need to compute P^{-1}.)

Solution: Since we have a basis consisting of eigenvectors, the matrix A is diagonalizable and we have $P^{-1}AP = D$, where

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

17. [**4 points**] Is the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix}$ invertible? If so, find A^{-1}.

Solution:

$$\left[\ A \mid I_3\ \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2R_1+R_3]{4R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 11 & 2 & 1 \\ 0 & 1 & 0 & 14 & 3 & 1 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & -4 & -1 \\ 0 & 1 & 0 & 14 & 3 & 1 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right].$$

Thus A is invertible and

$$A^{-1} = \begin{bmatrix} -17 & -4 & -1 \\ 14 & 3 & 1 \\ 10 & 2 & 1 \end{bmatrix}.$$

18. [**4 points**] Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 2 \\ 5 \\ 1 \\ -5 \end{bmatrix}$$

linearly dependent? If so, find a linear dependence relation.

Solution: We reduce the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & -5 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot, the corresponding system has nontrivial solutions. Thus the vectors are linearly dependent. The general solution to the above system is:

$$\begin{aligned} x_1 &= x_4 \\ x_2 &= -2x_4 \\ x_3 &- 3x_4 \\ x_4 &\text{ free} \end{aligned}$$

If $x_4 = 1$ (we could take any nonzero value for x_4), we have $x_1 = 1$, $x_2 = -2$ and $x_3 = -3$. Thus we have the linear dependence relation:

$$\vec{v}_1 - 2\vec{v}_2 - 3\vec{v}_3 + \vec{v}_4 = 0.$$

19. [**4 points**] Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

What is the dimension of this subspace?

Solution: The subspace spanned by the given vectors is the same as the column space of the matrix whose columns are these vectors. So we row reduce this matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 2 & -1 & -4 & 0 \end{bmatrix} \xrightarrow{-R_1+R_4} \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 4 & -4 & -9 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 4 & -4 & -9 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \xrightarrow{R_3+R_4} \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 4 & -4 & -9 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first, second and third columns are pivot columns, we know that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for the subspace spanned by the five vectors. The dimension of this subspace is therefore three.

20. [**4 points**] Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solve the matrix-vector equation $A\vec{x} = \vec{b}$. Write your answer in vector parametric form.

Solution: We have

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 1 & -3 & 1 \\ -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 3 & -3 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is thus:

$$x_1 = \frac{1}{3} + x_3$$
$$x_2 = \frac{1}{3} + x_3$$
$$x_3 \text{ free}$$

The solution set in vector parametric form is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

21. A closed economy consists of two sectors: telecommunications and services. To produce one unit, the telecommunications sector must consume 0.2 units from the telecommunications sector and 0.5 units from the services sector. On the other hand, to produce one unit, the services sector must consume 0.2 units from the telecommunications sector and 0.25 units from the services sector.

(a) [**1 point**] Give the consumption matrix C for this economy.

Solution:

$$C = \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.25 \end{bmatrix}.$$

(b) [**1 point**] Give the Leontief Input-Output Model production equation.

Solution: $\vec{x} = C\vec{x} + \vec{d}$ or $(I_2 - C)\vec{x} = \vec{d}$.

(c) [**3 points**] Find the production levels necessary to meet a final demand of $\vec{d} = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$

Solution: We solve the equation $(I_2 - C)\vec{x} = \vec{d}$ with

$$I_2 - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.25 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.2 \\ -0.5 & 0.75 \end{bmatrix}.$$

We have $\det(I_2 - C) = 0.6 - 0.1 = 0.5$ and so

$$(I_2 - C)^{-1} = \frac{1}{0.5}\begin{bmatrix} 0.75 & 0.2 \\ 0.5 & 0.8 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.4 \\ 1 & 1.6 \end{bmatrix}.$$

Therefore, the desired production levels are given by

$$\vec{x} = (I_2 - C)^{-1}\vec{d} = \begin{bmatrix} 1.5 & 0.4 \\ 1 & 1.6 \end{bmatrix}\begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 68 \\ 72 \end{bmatrix}.$$

22. In some country, company A has a monopoly on long distance telephone service. Company B has just entered the market (and thus has no clients initially). Each year, 40% of A's customers switch to B and 90% of B's clients switch to A.

(a) **[1 point]** Give the migration matrix M and the initial state vector $\vec{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) **[2 points]** Find the number of customers of A and B after two years as a fraction/percentage of the total number of customers.

Solution:

$$M^2\vec{x}_0 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .72 \\ .28 \end{bmatrix}$$

Therefore, after two years, A will have 72% of the customers and B will have 28% of the customers.

(c) **[4 points]** Find the steady-state vector. What fraction/percentage of the customers does A have in the long term? Remember to justify your answer.

Solution: To find the steady-state vector, we must find an eigenvector of eigenvalue 1. We row reduce:

$$\left[\begin{array}{c|c} M - I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -.4 & .9 & 0 \\ .4 & -.9 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -.4 & .9 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{10R_1} \left[\begin{array}{cc|c} -4 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is thus

$$\vec{x} = x_2 \begin{bmatrix} 9/4 \\ 1 \end{bmatrix}.$$

We choose the value of the free variable so that the sum of the entries of $\vec{x}$ is equal to one:

$$(9/4 + 1)x_2 = 1 \implies x_2 = 4/13.$$

Thus the steady-state vector is

$$\vec{x} = \begin{bmatrix} 9/13 \\ 4/13 \end{bmatrix}.$$

Since M is regular stochastic, the long term behaviour is given by the steady-state vector. Thus, in the long term, company A has 9/13 of the customers.