University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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Second Midterm Test Solutions – White Version

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Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	5	6	3	5	4	28
Grade							

QUESTION 1. **[5 pts]**

(a) Calculate the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 6 & 6 & 0 \\ 0 & 2 & 3 & -1 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$$

Solution: It is easiest to expand along the third column:

$$\begin{vmatrix} 2 & 6 & 6 & 0 \\ 0 & 2 & 3 & -1 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{vmatrix} = 6 \begin{vmatrix} 0 & 2 & -1 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix} + (-3) \begin{vmatrix} 2 & 6 & 0 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix}$$

Expanding the first 3×3 matrix along the first row:

$$\begin{vmatrix} 0 & 2 & -1 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = (-2)(7-3) + (-1)(7-15) = 0$$

Expanding the second 3×3 matrix along the first row:

$$\begin{vmatrix} 2 & 6 & 0 \\ 1 & 5 & 1 \\ 3 & 7 & 7 \end{vmatrix} = 2 \begin{vmatrix} 5 & 1 \\ 7 & 7 \end{vmatrix} + (-6) \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} = 2(35-7) + (-6)(7-3) = 32$$

Thus $\det A = 0 - 3(32) = -96$.

(b) Is A invertible? Justify your answer. If A is invertible, find the determinant of A^{-1}.

Solution: Yes, A is invertible since $\det A \neq 0$.

$$\det A^{-1} = \frac{1}{\det A} = -\frac{1}{96}.$$

QUESTION 2. [**5 pts**]

(a) If possible, find the inverse of the matrix A below.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -6 & -3 & 1 & 0 \\ 0 & 2 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -6 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[6R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -2 \\ 0 & -2 & 0 & -9 & 7 & 6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & \frac{9}{2} & \frac{-7}{2} & -3 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{3}{2} & 1 \\ 0 & 1 & 0 & \frac{9}{2} & \frac{-7}{2} & -3 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Thus, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{3}{2} & 1 \\ \frac{9}{2} & \frac{-7}{2} & -3 \\ -1 & 1 & 1 \end{bmatrix}.$$

(b) Suppose $\vec{b} \in \mathbb{R}^3$. How many solutions does the equation $A\vec{x} = \vec{b}$ have? Justify your answer.

Solution: Since A is invertible, the equation $A\vec{x} = \vec{b}$ has a unique solution given by $\vec{x} = A^{-1}\vec{b}$.

QUESTION 3. [**6 pts**] *The two parts of this question are independent of one another.*

(a) Suppose $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ -\frac{1}{2} & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -\frac{1}{3} & 5 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ and C is a 3×3 matrix such that

$$A^{-1}B(C^T)^{-1}A^2B^{-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Find $\det C$. *Hint:* Use the properties of determinants.

Solution: Using the fact that A and B are triangular, we easily compute that

$$\det A = (-1)\left(\frac{3}{2}\right)(2) = -3, \quad \det B = (6)\left(\frac{-1}{3}\right)\left(\frac{1}{2}\right) = -1.$$

Using the given equation, we see that:

$$\begin{aligned} \det(A^{-1}B(C^T)^{-1}A^2B^{-2}) &= 1(-2)(-1) = 2 \\ \implies \det A^{-1} \det B \det(C^T)^{-1} \det A^2 \det B^{-2} &= 2 \\ \implies \frac{1}{\det A} \det\ B \frac{1}{\det C}(\det A)^2 \frac{1}{(\det B)^2} &= 2 \\ \implies \frac{1}{\det C}(\det A)\frac{1}{\det B} &= 2 \\ \implies \frac{1}{\det C} &= 2\frac{\det B}{\det A} \\ \implies \det C = \frac{1}{2}\frac{\det A}{\det B} &= \frac{-3}{2(-1)} = \frac{3}{2} \end{aligned}$$

(b) Suppose A and B are 4×4 invertible matrices. Find a matrix X satisfying the matrix equation

$$A^{-1}\left(I + 2B(3X + I)^T\right)A = 2I + A.$$

Simplify your answer as much as possible.

Solution:

$$A^{-1}\left(I + 2B(3X + I)^T\right)A = 2I + A$$

$$\implies I + 2B(3X + I)^T = A(2I + A)A^{-1} = 2I + A$$

$$\implies 2B(3X + I)^T = I + A$$

$$\implies B(3X + I)^T = \frac{1}{2}(I + A)$$

$$\implies (3X + I)^T = \frac{1}{2}B^{-1}(I + A)$$

$$\implies 3X + I = \frac{1}{2}(B^{-1}(I + A))^T = \frac{1}{2}(I + A)^T(B^{-1})^T = \frac{1}{2}\left(I + A^T\right)(B^{-1})^T$$

$$\implies 3X = \frac{1}{2}\left(I + A^T\right)(B^{-1})^T - I$$

$$\implies X = \frac{1}{6}\left(I + A^T\right)(B^{-1})^T - \frac{1}{3}I$$

QUESTION 4. **[3 pts]** For each of the following subsets, state whether or not it is a subspace of $\mathbb{R}^n$ for the given n. Justify your answers.

(a) $H = \left\{ \begin{bmatrix} a \\ 1/a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \text{ and } a \neq 0 \right\}, \quad n = 3.$

Solution: No, H is not a subspace of $\mathbb{R}^3$ since it does not contain the zero vector.

(b) $W = \left\{ \begin{bmatrix} a \\ b \\ a+2b \\ 0 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}, \quad n = 4.$

Solution: Note that $\begin{bmatrix} a \\ b \\ a+2b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Thus

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Since all spans are subspaces, this implies that W is a subspace of $\mathbb{R}^4$.

(c) $V = \left\{ \begin{bmatrix} x \\ y \\ xy \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}, \quad n = 3.$

Solution: No, V is not a subspace of $\mathbb{R}^3$ since, for instance,

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in V \quad \text{but} \quad (-1)\vec{v} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \notin V,$$

because $-1 \neq (-1)(-1)$.

QUESTION 5. [**5 pts**] Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 2 \\ 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 2 \end{bmatrix}.$$

(a) Find a basis for Nul A.

Solution: To solve the homogeneous system $A\vec{x} = \vec{0}$, we row reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & 3 & 0 & -4 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_4]{-R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Switching to equation notation gives

$$\begin{aligned} x_1 &= -3x_3 + 4x_5 \\ x_2 &= x_3 - 2x_5 \\ x_3 &\text{ free} \\ x_4 &= 2x_5 \\ x_5 &\text{ free} \end{aligned}$$

Therefore the null space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 + 4x_5 \\ x_3 - 2x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_3, x_5 \in \mathbb{R}.$$

So a basis of Nul A is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for Col A. What is the rank of A? Here A is the same matrix as in the previous part of this question.

Solution: By the row reduction above, the pivot columns of A are columns 1, 2 and 4. Thus, a basis of $\operatorname{Col} A$ is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Therefore, $\operatorname{rank} A = 3$.

QUESTION 6. **[4 pts]** Consider an economy divided into 2 sectors: Services and Transportation. In order to produce one unit of output, Services must consume 0.6 units from its own sector and 0.2 units from Transportation. On the other hand, to produce one unit of output, Transportation must consume 0.1 units from its own sector and 0.3 units from Services.

(a) Give the consumption matrix C for this economy.

Solution: If Services is the first sector and Transportation is the second, then

$$C = \begin{bmatrix} .6 & .3 \\ .2 & .1 \end{bmatrix}.$$

If Transportation is the first sector and Services is the second sector, then

$$C = \begin{bmatrix} .1 & .2 \\ .3 & .6 \end{bmatrix}$$

(b) Find the intermediate demand if Services wants to produce 15 units and Transportation wants to produce 20 units.

Solution: The intermediate demand is

$$15 \begin{bmatrix} .6 \\ .2 \end{bmatrix} + 20 \begin{bmatrix} .3 \\ .1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}.$$

Thus the intermediate demand is 15 units from Services and 5 units from Transportation.

(c) Determine the production levels needed to meet a final demand of 24 units from Services and 12 units from Transportation.

Solution: We need to solve the Leontief equation

$$(I - C)\vec{x} = \begin{bmatrix} 24 \\ 12 \end{bmatrix}.$$

We have

$$I - C = \begin{bmatrix} .4 & -.3 \\ -.2 & .9 \end{bmatrix} \implies (\mathrm{I} - C)^{-1} = \frac{1}{.3} \begin{bmatrix} .9 & .3 \\ .2 & .4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}.$$

Therefore

$$\vec{x} = (\mathrm{I} - C)^{-1} \begin{bmatrix} 24 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 24 \\ 12 \end{bmatrix} = \begin{bmatrix} 84 \\ 32 \end{bmatrix}$$

and so Services needs to produce 84 units and Transportation needs to produce 32 units.

Alternate method: Row reduce the augmented matrix $\left[\, I - C \,\middle|\, \vec{d} \,\right]$.