University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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First Midterm Test (White Version) – Solutions

10 February 2012

Surname ____________________ First Name ____________________

Student # ____________________ DGD (1–4) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	3	6	5	4	4	26
Grade							

Question 1. [**4 pts**] Is the following linear system consistent or inconsistent? (If it is consistent, you do *not* need to completely solve the system.)

$$\begin{aligned} 3x_2 &= 5 + x_3 \\ x_1 + 2x_3 + 2 &= 2 - x_2 \\ 3x_1 + 6x_2 + 5x_3 &= 3 \\ -x_1 - 7x_2 + 10x_3 + 10 &= 10x_3 \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcr} & & 3x_2 & - & x_3 & = & 5 \\ x_1 & + & x_2 & + & 2x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 5x_3 & = & 3 \\ -x_1 & - & 7x_2 & & & = & -10 \end{array}$$

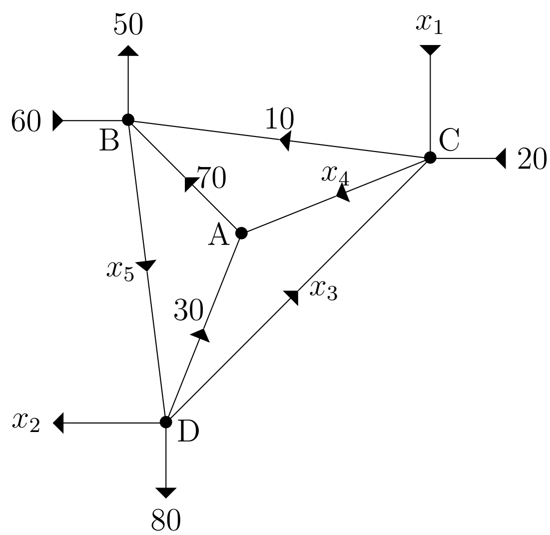
Then write down the augmented matrix and row reduce to echelon form:

$$\left[\begin{array}{rrr|r} 0 & 3 & -1 & 5 \\ 1 & 1 & 2 & 0 \\ 3 & 6 & 5 & 3 \\ -1 & -7 & 0 & -10 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & 3 & -1 & 5 \\ 3 & 6 & 5 & 3 \\ -1 & -7 & 0 & -10 \end{array}\right] \xrightarrow[R_1+R_4]{-3R_1+R_3} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & 3 & -1 & 5 \\ 0 & 3 & -1 & 3 \\ 0 & -6 & 2 & -10 \end{array}\right]$$

$$\xrightarrow[2R_2+R_4]{-R_2+R_3} \left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the rightmost column is a pivot column, the system is inconsistent.

QUESTION 2. **[3 pts]** Write down a system of equations describing the flow in the following network. Include all relevant equations. You do *not* need to solve the system.



Solution: Setting the total flow in equal to the flow out at each intersection and for the overall system as a whole, we get the following linear system.

Intersection	**Flow in**		**Flow out**
A	$30 + x_4$	$=$	70
B	$60 + 70 + 10$	$=$	$50 + x_5$
C	$20 + x_1 + x_3$	$=$	$10 + x_4$
D	x_5	$=$	$80 + 30 + x_2 + x_3$
Overall	$60 + 20 + x_1$	$=$	$50 + 80 + x_2$

QUESTION 3. [**6 pts**] Suppose

$$A = \begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & 5 & 5 \\ -6 & -2 & -14 & -16 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(a) Find the general solution to the matrix-vector equation $A\vec{x} = \vec{b}$. Write your answer in *vector parametric form.*

Solution: We row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 1 \\ 2 & 1 & 5 & 5 & -1 \\ -6 & -2 & -14 & -16 & 2 \end{array}\right] \xrightarrow[6R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 1 \\ 0 & 3 & 3 & -3 & -3 \\ 0 & -8 & -8 & 8 & 8 \end{array}\right] \xrightarrow{\frac{8}{3}R_2+R_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 1 \\ 0 & 3 & 3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Switching to equation notation and solving for the basic variables in terms of the free variables, the general solution is:

$$\begin{aligned} x_1 &= -2x_3 - 3x_4 \\ x_2 &= -1 - x_3 + x_4 \\ x_3, x_4 &\text{ free} \end{aligned}$$

Switching to vector parametric form gives:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -1 - x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$

(b) Give the general solution to the corresponding homogeneous equation $A\vec{x} = \vec{0}$. Again, write your answer in vector parametric form.

Solution: The general solution to the corresponding homogeneous equation is

$$\vec{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$

Question 4. [**5 pts**] Are the vectors

$$\vec{a}_1 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \text{and} \quad \vec{a}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

linearly dependent or linearly independent? If they are linearly dependent, find a linear dependence relation.

Solution: The question is equivalent to asking if the vector equation

$$x_1 \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution. Therefore, we write down the corresponding augmented matrix (with the vectors in question as the columns of the coefficient matrix) and row reduce to echelon form:

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ -1 & 4 & 5 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] \xrightarrow{\frac{-1}{2}R_1+R_2} \left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the system is consistent (since the last column is not a pivot column) and the general solution has a free variable (since the third column is not a pivot column). Thus the vectors are linearly dependent.

To find a linear dependence relation, we need to find a nontrivial solution. We continue reducing to reduced echelon form

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Returning to equation notation and solving for the basic variables in terms of the free variables, we obtain:

$$\begin{array}{rl} x_1 - x_3 & = 0 \\ x_2 + x_3 & = 0 \\ x_3 & \text{free} \end{array} \quad \rightsquigarrow \quad \begin{array}{rl} x_1 & = x_3 \\ x_2 & = -x_3 \\ x_3 & \text{free} \end{array}$$

Since we only need one linear dependence relation, we can take any nontrivial solution. For instance, taking $x_3 = 1$, we have $x_1 = 1$ and $x_2 = -1$. This gives the linear dependence relation

$$\vec{a}_1 - \vec{a}_2 + \vec{a}_3 = \vec{0}.$$

QUESTION 5. [**4 pts**] Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 9 \end{bmatrix}.$$

Is $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Solution: We need to know if the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

has a solution. We write down the corresponding augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ -2 & 0 & 1 & 3 \\ 0 & 4 & 1 & 9 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 4 & 1 & 9 \end{array}\right] \xrightarrow{-4R_2+R_4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3+R_4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the system is consistent, and so the answer is YES.

QUESTION 6. [**4 pts**] For which values of p and q does the system

$$\begin{array}{rcrcl} x_1 & + & 4x_2 & = & p \\ -2x_1 & + & qx_2 & = & 6 \end{array}$$

have *exactly one* solution. Your answer should include *all* values of p and q for which the system has exactly one solution.

Solution: We row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 4 & p \\ -2 & q & 6 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{cc|c} 1 & 4 & p \\ 0 & 8+q & 2p+6 \end{array}\right]$$

In order for the system to have exactly one solution, the system must be consistent and have no free variables. This is the case exactly when we have a pivot in each column of the coefficient matrix. Therefore the system has exactly one solution when $q \neq -8$ and p is any real number.