University of Ottawa

Department of Mathematics and Statistics

MAT 1302: Mathematical Methods II

Instructor: Alistair Savage

Final Exam

23 April 2012

Surname ____________________ First Name ____________________

Student # ____________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 16 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 17 through 24 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) The number of points available for each question is indicated in square brackets.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–16	17	18	19	20	21	22	23	24	Total
Maximum	27	4	4	5	4	4	9	4	6	67
Grade										

Part A: Answer Only Questions

For Questions 1–16, only your final answer will be considered for marks. Write your final answer(s) in the space(s) provided.

1. [**1 pt**] What is the size of a matrix with characteristic polynomial $\lambda^6 - 5\lambda^4 + 3\lambda^3 - 2\lambda + 7$?

Answer:________________________

2. [**2 pts**] Find a vector equation of the line passing through the points $(1, -2, 1)$ and $(3, 5, 4)$.

Answer:________________________

3. [**2 pts**] Give a basis for Nul A, where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Answer:________________________

4. [**2 pts**] Suppose

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Compute AB and $B^T - A$.

Answer: $AB =$ ________________________, $B^T - A =$ ________________________

5. [**1 pt**] Suppose A is a 7×9 matrix with $\dim \operatorname{Nul} A = 3$. What is the rank of A?

Answer: ________________________

6. [**2 pts**] If the complex number

$$\frac{2+3i}{1+4i}$$

is written in the form $a + bi$, with $a, b \in \mathbb{R}$, what are a and b?

Answer: $a =$ ______________, $b =$ ______________

7. [**2 pts**] If $z = 2 - i$ and $w = 1 + 3i$, what are $2z + w$ and $z\bar{w}$? Write your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Answer: $2z + w =$ ___________________, $z\bar{w} =$ ___________________

8. [**1 pt**] Write the Leontief Input-Output Model production equation.

Answer: ________________________

9. [**1 pt**] Suppose an economy is divided into three sectors: Technology, Agriculture, and Textiles. In order to produce one unit, Technology must consume 0.4 units from Technology, 0.3 units from Agriculture, and nothing from Textiles. To product one unit, Agriculture must consume 0.1 units from Technology, 0.3 units from Agriculture, and 0.2 units from Textiles. In order to produce one unit, Textiles must consume 0.4 units from Technology, 0.2 units from Agriculture, and 0.2 units from Textiles. Write the consumption matrix for this economy.

Answer: ________________________

10. [**2 pts**] Which of the following statements are true for **every** invertible $n \times n$ matrix A? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) For **any** $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution.
(b) The matrix AB is invertible for any $n \times n$ matrix B.
(c) The matrix A^T is invertible.
(d) The matrix A has zero as an eigenvalue.
(e) The determinant of A is equal to one.

Answer:________________________

11. [**2 pts**] Which of the following statements are true? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) It is possible for a system of linear equations to have exactly two solutions.
(b) Every homogeneous system is consistent.
(c) The number 5 is a complex number.
(d) A set of 5 vectors in $\mathbb{R}^4$ must be linearly dependent.
(e) The space $\mathbb{R}^3$ only has one basis.

Answer:________________________

12. [**2 pts**] Which of the the following statements are true? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) Eigenvalues can never be zero.
(b) Eigenvectors can never be the zero vector.
(c) If $\vec{x}$ is an eigenvector of a matrix A, then $2\vec{x}$ is also an eigenvector of A.
(d) The maximum possible rank of 5×7 matrix is 5.
(e) If a set of three vectors is linearly dependent, then one of the vectors is a multiple of another.

Answer:________________________

13. [**2 pts**] Which of the following subsets are subspaces of $\mathbb{R}^n$ for the given n? Note that more than one subset may be a subspace. You should indicate **all** the subspaces. (You will lose points for including sets that are not subspaces.)

(a) $\{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y = 0 \text{ and } 2x + y - z = 0\}$, $n = 3$.

(b) The set of all $\vec{b} \in \mathbb{R}^2$ such that the equation $A\vec{x} = \vec{b}$ has a solution; $n = 2$. Here A is a fixed 2×3 matrix.

(c) $\{(x, x^2) \mid x \in \mathbb{R}\}$, $n = 2$.

(d) $\{(1, x, y, z) \mid x, y, z \in \mathbb{R}\}$, $n = 4$.

(e) The span of a set of five vectors in $\mathbb{R}^3$, $n = 3$.

Answer:________________________

14. [**1 pt**] Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{12} & a_{11} & a_{13} & a_{14} + 3a_{13} \\ a_{22} & a_{21} & a_{23} & a_{24} + 3a_{23} \\ a_{32} & a_{31} & a_{33} & a_{34} + 3a_{33} \\ a_{42} & a_{41} & a_{43} & a_{44} + 3a_{43} \end{bmatrix}.$$

If $\det A = 3$, what is $\det B$?

Answer: $\det B =$________________________

15. **[2 pts]** Suppose A is a 3×3 matrix with $\det A = 2$. What are the determinants of $2A^T$ and A^{-1}?

Answer: $\det 2A^T =$______________, $\det A^{-1} =$______________

16. **[2 pts]** For which values of c (if any) does the linear system with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & c & 6 \end{array}\right]$$

have infinitely many solutions?

Answer:________________________

Part B: Long Answer Questions

For Questions 17–24, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

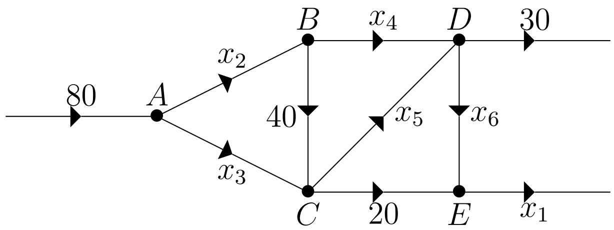
17. [**4 pts**] Find the general solution to the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 5x_2 & - & x_3 & - & 3x_4 & = & 1 \\ x_1 & - & 4x_2 & + & 2x_3 & + & 3x_4 & = & -8 \\ x_1 & - & 7x_2 & + & 3x_3 & + & 5x_4 & = & -11 \end{array}$$

18. [**4 pts**] Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 5 & 0 & 3 & -2 \\ 4 & 6 & 0 & 1 \end{bmatrix}.$$

19. Consider network depicted below.



(a) **[2 pts]** Write a system of linear equations describing the flow in this network. You do **not** need to solve the system.

(b) [**2 pts**] The reduced echelon form of the augmented matrix of the system from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 50 \\ 0 & 1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Write the general solution for the flow in the network. Indicate the ranges of possible values of any free variables (that is, give the minimum and maximum values of any free variables).

(c) [**1 pt**] What are the minimum and maximum flows along the branch labeled by x_2?

20.

(a) **[3 pts]** Find all values of t such that the three vectors given below form a linearly independent set. Justify your answer.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3t \end{bmatrix}$$

(b) **[1 pt]** If $t = 1$, can **every** $\vec{b} \in \mathbb{R}^3$ be written as a linear combination of the three vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$? Justify your answer.

21. [**4 pts**] If possible, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 2 & -1 \\ -2 & 0 & -5 \end{bmatrix}.$$

22.

(a) [**3 pts**] Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 4 & 3 \end{bmatrix}.$$

List the eigenvalues of A, together with their multiplicities.

(b) [**5 pts**] The matrix

$$B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$$

has eigenvalues 2 (with multiplicity 2) and 1 (with multiplicity 1). For each eigenvalue, find a basis of the corresponding eigenspace.

(c) **[1 pt]** If possible, diagonalize the matrix B of part (b). That is, find a diagonal matrix D and an invertible matrix P such that $B = PDP^{-1}$. (Note: You do *not* need to find P^{-1}.)

23.

(a) [**3 pts**] Find a basis for the subspace

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -5 \\ 14 \end{bmatrix} \right\}.$$

(b) [**1 pt**] What is the dimension of the subspace given in part (a)?

24. *Prince Markov*, the leading brand of vodka, currently has 80% of the vodka market. Its competitor, *Kolmogorov-Smirnov*, has 20%. Six months after aggressive marketing campaigns by both companies, 10% of *Prince Markov* drinkers switched to *Kolmogorov-Smirnov*, and 5% of *Kolmogorov-Smirnov* drinkers switched to *Prince Markov*.

(a) **[1 pt]** Write the transition (migration) matrix P for this problem.

(b) **[1 pt]** What are the market shares of *Prince Markov* and *Kolmogorov-Smirnov*, respectively, after the marketing campaign?

(c) **[4 pts]** If these trends continue, what will be the long term market shares of the two brands? (Do not forget to justify **why** your method gives the long term behaviour.) Will *Prince Markov* eventually lose market dominance and have less than 50% of the market share?

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