University of Ottawa

Department of Mathematics and Statistics

MAT 1302: Mathematical Methods II

Instructor: Alistair Savage

Final Exam

23 April 2012

Surname ____________________ First Name ____________________

Student # ____________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 16 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 17 through 24 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) The number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–16	17	18	19	20	21	22	23	24	Total
Maximum	27	4	4	5	4	4	9	4	6	67
Grade										

Part A: Answer Only Questions

For Questions 1–16, only your final answer will be considered for marks. Write your final answer(s) in the space(s) provided.

1. [**1 pt**] What is the size of a matrix with characteristic polynomial $\lambda^6 - 5\lambda^4 + 3\lambda^3 - 2\lambda + 7$?

   Answer: 6×6

2. [**2 pts**] Find a vector equation of the line passing through the points $(1, -2, 1)$ and $(3, 5, 4)$.

   Answer: A vector equation of the line is $(1, -2, 1) + t(-2, -7, -3)$, $t \in \mathbb{R}$. (Other answers are possible.)

3. [**2 pts**] Give a basis for $\operatorname{Nul} A$, where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   Answer: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. [**2 pts**] Suppose

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Compute AB and $B^T - A$.

   Answer: $AB = \begin{bmatrix} 6 & 3 \\ 0 & 5 \end{bmatrix}$ and $B^T - A = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

5. [**1 pt**] Suppose A is a 7×9 matrix with $\dim \operatorname{Nul} A = 3$. What is the rank of A?

   Answer: 6

6. [**2 pts**] If the complex number

$$\frac{2 + 3i}{1 + 4i}$$

is written in the form $a + bi$, with $a, b \in \mathbb{R}$, what are a and b?

   Answer: $a = \frac{14}{17}$, $b = \frac{-5}{17}$

7. [**2 pts**] If $z = 2 - i$ and $w = 1 + 3i$, what are $2z + w$ and $z\bar{w}$? Write your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

Answer: $2z + w = 5 + i$, and $z\bar{w} = -1 - 7i$.

8. **[1 pt]** Write the Leontief Input-Output Model production equation.

Answer: $\vec{x} = C\vec{x} + \vec{d}$ or $(I - C)\vec{x} = \vec{d}$.

9. **[1 pt]** Suppose an economy is divided into three sectors: Technology, Agriculture, and Textiles. In order to produce one unit, Technology must consume 0.4 units from Technology, 0.3 units from Agriculture, and nothing from Textiles. To product one unit, Agriculture must consume 0.1 units from Technology, 0.3 units from Agriculture, and 0.2 units from Textiles. In order to produce one unit, Textiles must consume 0.4 units from Technology, 0.2 units from Agriculture, and 0.2 units from Textiles. Write the consumption matrix for this economy.

Answer: $C = \begin{bmatrix} 0.4 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix}$

10. **[2 pts]** Which of the following statements are true for **every** invertible $n \times n$ matrix A? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) For **any** $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has exactly one solution.
(b) The matrix AB is invertible for any $n \times n$ matrix B.
(c) The matrix A^T is invertible.
(d) The matrix A has zero as an eigenvalue.
(e) The determinant of A is equal to one.

Answer: (a), (c)

11. **[2 pts]** Which of the following statements are true? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) It is possible for a system of linear equations to have exactly two solutions.
(b) Every homogeneous system is consistent.
(c) The number 5 is a complex number.
(d) A set of 5 vectors in $\mathbb{R}^4$ must be linearly dependent.
(e) The space $\mathbb{R}^3$ only has one basis.

Answer: (b), (c), (d)

12. **[2 pts]** Which of the the following statements are true? Note that more than one statement may be true. You should indicate **all** the true statements. (You will lose points for indicating that false statements are true.)

(a) Eigenvalues can never be zero.
(b) Eigenvectors can never be the zero vector.
(c) If $\vec{x}$ is an eigenvector of a matrix A, then $2\vec{x}$ is also an eigenvector of A.
(d) The maximum possible rank of 5×7 matrix is 5.

(e) If a set of three vectors is linearly dependent, then one of the vectors is a multiple of another.

Answer: (b), (c), (d)

13. [**2 pts**] Which of the following subsets are subspaces of $\mathbb{R}^n$ for the given n? Note that more than one subset may be a subspace. You should indicate **all** the subspaces. (You will lose points for including sets that are not subspaces.)

(a) $\{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y = 0 \text{ and } 2x + y - z = 0\}$, $n = 3$.
(b) The set of all $\vec{b} \in \mathbb{R}^2$ such that the equation $A\vec{x} = \vec{b}$ has a solution; $n = 2$. Here A is a fixed 2×3 matrix.
(c) $\{(x, x^2) \mid x \in \mathbb{R}\}$, $n = 2$.
(d) $\{(1, x, y, z) \mid x, y, z \in \mathbb{R}\}$, $n = 4$.
(e) The span of a set of five vectors in $\mathbb{R}^3$, $n = 3$.

Answer: (a), (b), (e)

14. [**1 pt**] Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{12} & a_{11} & a_{13} & a_{14} + 3a_{13} \\ a_{22} & a_{21} & a_{23} & a_{24} + 3a_{23} \\ a_{32} & a_{31} & a_{33} & a_{34} + 3a_{33} \\ a_{42} & a_{41} & a_{43} & a_{44} + 3a_{43} \end{bmatrix}.$$

If $\det A = 3$, what is $\det B$?

Answer: -3

15. [**2 pts**] Suppose A is a 3×3 matrix with $\det A = 2$. What are the determinants of $2A^T$ and A^{-1}?

Answer: $\det 2A^T = 16$, $\det A^{-1} = \frac{1}{2}$.

16. [**2 pts**] For which values of c (if any) does the linear system with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & c & 6 \end{array}\right]$$

have infinitely many solutions?

Answer: $c = 6$

Part B: Long Answer Questions

For Questions 17–24, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

17. [**4 pts**] Find the general solution to the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 5x_2 & - & x_3 & - & 3x_4 & = & 1 \\ x_1 & - & 4x_2 & + & 2x_3 & + & 3x_4 & = & -8 \\ x_1 & - & 7x_2 & + & 3x_3 & + & 5x_4 & = & -11 \end{array}$$

Solution: The augmented matrix for this system is:

$$\left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 1 & -4 & 2 & 3 & -8 \\ 1 & -7 & 3 & 5 & -11 \end{array}\right]$$

Row reduction, as follows, leads to row reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 1 & -4 & 2 & 3 & -8 \\ 1 & -7 & 3 & 5 & -11 \end{array}\right] \xrightarrow[-R_1+R_2]{-R_1+R_3} \left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 0 & -9 & 3 & 6 & -9 \\ 0 & -12 & 4 & 8 & -12 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_2]{\frac{1}{4}R_3} \left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 0 & 3 & -1 & -2 & 3 \\ 0 & -3 & 1 & 2 & -3 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 0 & 3 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 5 & -1 & -3 & 1 \\ 0 & 1 & -1/3 & -2/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-5R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2/3 & 1/3 & -4 \\ 0 & 1 & -1/3 & -2/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

We can then write down the general solution:

$$x_1 = -4 - \frac{2}{3}x_3 - \frac{1}{3}x_4$$

$$x_2 = 1 + \frac{1}{3}x_3 + \frac{2}{3}x_4$$

$$x_3, x_4 \text{ free}$$

18. [**4 pts**] Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 5 & 0 & 3 & -2 \\ 4 & 6 & 0 & 1 \end{bmatrix}.$$

Solution: It is easiest to expand along the second row (or the third column):

$$\begin{vmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 5 & 0 & 3 & -2 \\ 4 & 6 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} 2 & 0 & 1 \\ 5 & 3 & -2 \\ 4 & 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & -3 & 1 \\ 5 & 0 & -2 \\ 4 & 6 & 1 \end{vmatrix}$$

Expanding the first 3×3 matrix along the second column:

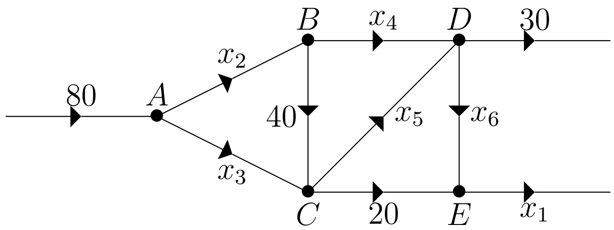
$$\begin{vmatrix} 2 & 0 & 1 \\ 5 & 3 & -2 \\ 4 & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 3(2-4) = -6$$

Expanding the second 3×3 matrix along the second row:

$$\begin{vmatrix} 2 & -3 & 1 \\ 5 & 0 & -2 \\ 4 & 6 & 1 \end{vmatrix} = -5 \begin{vmatrix} -3 & 1 \\ 6 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} = -5(-3-6) + 2(12+12) = 45 + 48 = 93$$

Thus $\det A = (-1)(-6) + 93 = 99$.

19. Consider network depicted below.



(a) **[2 pts]** Write a system of linear equations describing the flow in this network. You do **not** need to solve the system.

Solution: For each vertex/intersection, we set the total flow in equal to the total flow out. We also set the total overall flow in equal to the total overall flow out. This yields the following linear system.

$$
\begin{array}{lrcl}
A: & 80 & = & x_2 + x_3 \\
B: & x_2 & = & x_4 + 40 \\
C: & x_3 + 40 & = & x_5 + 20 \\
D: & x_4 + x_5 & = & x_6 + 30 \\
E: & x_6 + 20 & = & x_1 \\
\text{Overall:} & 80 & = & x_1 + 30
\end{array}
$$

(b) **[2 pts]** The reduced echelon form of the augmented matrix of the system from part (a) is

$$
\left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 50 \\
0 & 1 & 0 & 0 & 1 & 0 & 100 \\
0 & 0 & 1 & 0 & -1 & 0 & -20 \\
0 & 0 & 0 & 1 & 1 & 0 & 60 \\
0 & 0 & 0 & 0 & 0 & 1 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

Write the general solution for the flow in the network. Indicate the ranges of possible values of any free variables (that is, give the minimum and maximum values of any free variables).

Solution: The general solution is:

$$
\begin{aligned}
x_1 &= 50 \\
x_2 &= 100 - x_5 \\
x_3 &= x_5 - 20 \\
x_4 &= 60 - x_5 \\
x_6 &= 30 \\
x_5 & \text{ free}
\end{aligned}
$$

Since x_2, x_3, and x_5 must all be positive, we have the constraints $x_5 \leq 100$, $x_5 \geq 20$, and $x_5 \leq 60$. Thus, the range of possible values for the free variable x_5 is $20 \leq x_5 \leq 60$.

(c) **[1 pt]** What are the minimum and maximum flows along the branch labeled by x_2?

Solution: Since $20 \leq x_5 \leq 60$, the minimum flow is 40 and the maximum is 80.

20.

(a) **[3 pts]** Find all values of t such that the three vectors given below form a linearly independent set. Justify your answer.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3t \end{bmatrix}$$

Solution: We write write the vectors as the columns of a matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ t & 1 & 1 & 0 \\ 1 & 1 & 3t & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-tR_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1-t & 0 \\ 0 & 1 & 3t-1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1-t & 0 \\ 0 & 0 & 4t-2 & 0 \end{array}\right]$$

The set of vectors is linearly independent if and only if we only have the trivial solution, i.e. no free variables. This happens when

$$4t - 2 \neq 0 \iff 4t \neq 2 \iff t \neq \frac{1}{2}.$$

(b) **[1 pt]** If $t = 1$, can **every** $\vec{b} \in \mathbb{R}^3$ be written as a linear combination of the three vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$? Justify your answer.

Solution: For $t = 1$, the vectors are linearly independent and so A has three pivots positions. Thus $A\vec{x} = \vec{b}$ is consistent for any vector $\vec{b} \in \mathbb{R}^3$. Therefore $\vec{b}$ is a linear combination of the columns of A, and so the answer is YES.

21. [**4 pts**] If possible, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 2 & -1 \\ -2 & 0 & -5 \end{bmatrix}.$$

Solution: We form the augmented matrix and row reduce:

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2R_1+R_3]{3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1]{5R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 0 & 2 \\ 0 & 2 & 0 & 13 & 1 & 5 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[-R_3]{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 0 & 2 \\ 0 & 1 & 0 & \frac{13}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & -2 & 0 & -1 \end{array}\right]$$

Thus A is invertible and

$$A^{-1} = \begin{bmatrix} 5 & 0 & 2 \\ \frac{13}{2} & \frac{1}{2} & \frac{5}{2} \\ -2 & 0 & -1 \end{bmatrix}.$$

22.

(a) **[3 pts]** Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 4 & 3 \end{bmatrix}.$$

List the eigenvalues of A, together with their multiplicities.

Solution: The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 5-\lambda & 2 \\ 0 & 4 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)\big((5-\lambda)(3-\lambda) - 8\big) \\ &= (1-\lambda)^2(7-\lambda). \end{aligned}$$

Thus, the eigenvalues are $\lambda = 1$ (with multiplicity 2) and $\lambda = 7$ (with multiplicity 1).

(b) **[5 pts]** The matrix

$$B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$$

has eigenvalues 2 (with multiplicity 2) and 1 (with multiplicity 1). For each eigenvalue, find a basis of the corresponding eigenspace.

Solution: For $\lambda = 1$, we row reduce the matrix $\left[\begin{array}{c|c} B - I & 0 \end{array}\right]$ to find the general solution to the homogenous system $(B - I)\vec{x} = 0$.

$$\left[\begin{array}{ccc|c} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

Therefore, a basis of the eigenspace is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = 2$, we row reduce the matrix $\left[\begin{array}{c|c} B - 2I & 0 \end{array}\right]$ to find the general solution to the homogenous system $(B - 2I)\vec{x} = 0$.

$$\left[\begin{array}{ccc|c} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}.$$

Therefore, a basis of the eigenspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) **[1 pt]** If possible, diagonalize the matrix B of part (b). That is, find a diagonal matrix D and an invertible matrix P such that $B = PDP^{-1}$. (Note: You do *not* need to find P^{-1}.)

Solution: We can take

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}; \quad \text{or}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & -3 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}; \quad \text{or}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & -2 & -3 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \quad \text{etc.}$$

23.

(a) [**3 pts**] Find a basis for the subspace

$$\text{Span}\left\{\begin{bmatrix}0\\1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\-1\\1\\-2\end{bmatrix},\begin{bmatrix}1\\2\\-2\\4\end{bmatrix},\begin{bmatrix}-5\\3\\-2\\6\end{bmatrix},\begin{bmatrix}2\\7\\-5\\14\end{bmatrix}\right\}.$$

Solution: We row reduce to echelon form:

$$\begin{bmatrix}0&0&1&-5&2\\1&-1&2&3&7\\-1&1&-2&-2&-5\\2&-2&4&6&14\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}1&-1&2&3&7\\0&0&1&-5&2\\-1&1&-2&-2&-5\\2&-2&4&6&14\end{bmatrix}\xrightarrow[-2R_1+R_4]{R_1+R_3}\begin{bmatrix}1&-1&2&3&7\\0&0&1&-5&2\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}.$$

Since the pivot columns are the first, third and fourth columns, a basis for the subspace is

$$\left\{\begin{bmatrix}0\\1\\-1\\2\end{bmatrix},\begin{bmatrix}1\\2\\-2\\4\end{bmatrix},\begin{bmatrix}-5\\3\\-2\\6\end{bmatrix}\right\}.$$

(b) [**1 pt**] What is the dimension of the subspace given in part (a)?

Solution: Because the basis has three elements, the dimension of the subspace is three.

24. *Prince Markov*, the leading brand of vodka, currently has 80% of the vodka market. Its competitor, *Kolmogorov-Smirnov*, has 20%. Six months after aggressive marketing campaigns by both companies, 10% of *Prince Markov* drinkers switched to *Kolmogorov-Smirnov*, and 5% of *Kolmogorov-Smirnov* drinkers switched to *Prince Markov*.

(a) **[1 pt]** Write the transition (migration) matrix P for this problem.

Solution: This is a two state system. If M represents the proportion of drinkers of *Prince Markov* and K, the proportion of *Kolmogorov-Smirnov* drinkers, then the transition matrix is

$$P = \begin{array}{c} \\ M \\ K \end{array} \begin{array}{c} \begin{array}{cc} M & K \end{array} \\ \begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \end{array}$$

(b) **[1 pt]** What are the market shares of *Prince Markov* and *Kolmogorov-Smirnov*, respectively, after the marketing campaign?

Solution: The initial market distribution is $(0.8, 0.2)$. After the advertising campaign, market share is:

$$\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.73 \\ 0.27 \end{bmatrix}$$

(c) **[4 pts]** If these trends continue, what will be the long term market shares of the two brands? (Do not forget to justify **why** your method gives the long term behaviour.) Will *Prince Markov* eventually lose market dominance and have less than 50% of the market share?

Solution: Since the transition matrix P is regular stochastic, the long term behaviour is given by the steady-state vector. We find the steady-state vector by solving $(P - I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{c|c} P - I & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.1 & 0.05 & 0 \\ 0.1 & -0.05 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$M = 0.5K$$
$$K \text{ free}$$

In vector parametric form, we have

$$\begin{bmatrix} M \\ K \end{bmatrix} = \begin{bmatrix} 0.5K \\ K \end{bmatrix} = K \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad K \in \mathbb{R}.$$

Since we want a steady-state vector whose elements sum to 1 (because this is a vector of probabilities!), we solve

$$0.5K + K = 1 \implies 1.5K = 1 \implies K = \frac{2}{3}.$$

Thus, the final market share distribution is $\begin{bmatrix}1/3\\2/3\end{bmatrix}$. Final market share for *Prince Markov* is 1/3, which is less than 50%. Thus, *Kolmogorov-Smirnov* will become the leading brand.