Lilac

MAT 1302 D – Test 3 – March 29th ,Winter 2011

Instructor: Termeh Kousha

[Print your FAMILY NAME in CAPITAL letters]

Name: Marking Scheme and Solutions.

Student Number:____________________________

Signature:____________________________

Make sure your cell phone is off before starting...

Instructions: This exam consists of 5 questions in 8 pages. The marks for each question are as listed with the question itself. The exam is out of **30**. No calculators or other electronic aids allowed. No notes, books or other papers allowed.
Write all your answers in **non-erasable pen**. If you make a mistake just scratch it out and continue. You may use the back of pages for answer of questions.

GOOD LUCK!

1. [3 points] Determine if the vectors are linearly independent. Justify your answer.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 2 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Basic

Three Pivots → so the system is L.I

2. [3 points] **Just by inspection**, determine whether the vectors are linearly dependent. Justify your answer clearly.

a. $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$

L.D since 3 vectors of size 2x1.

(# of vectors is greater than # entries in each column]

b. $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

L.D Because of the zero vector.

c. $\begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix}.$

L.I, 2 vectors which are not collinear.

3. [4 points] Find the determinant of $A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \\ 2 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ + \\ - \\ + \end{array}$

$$= 2 \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 1 & 1 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & -3 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{vmatrix}$$

$$= 2(1)[-9+12] + 0$$

$$= 2(3) = 6$$

For question 4, you have to show ALL the details. If you claim something, you need to prove it.

4.(a) [4 points] Show that if A and B are similar, then they have the same eigenvalues.

1 point A similar to B, $\Rightarrow$ there exists an invertible matrix P such that 0.5 point

$A = PBP^{-1}$

$(\det(P)\det(P^{-1}) = 1)$

0.5 point $\rightarrow$ we show they have the same characteristic equation
(so they have the same eigenvalue)

1 point $\det|A - \lambda I| = \det|PBP^{-1} - \lambda I| = \det(PBP^{-1} - \lambda PIP^{-1})$

$= \det(P(B - \lambda I)P^{-1}) = \det(P)\det(B - \lambda I)\det(P)^{-1}$

$= \det(B - \lambda I)$

(b) [4 points] Show that λ is an eigenvalue of a matrix A if and only if λ is an eigenvalue of A^T.

assume λ is an eigenvalue of A

$\Downarrow$

1 $(A - I\lambda)\vec{x} = \vec{0}$ has a non-trivial solution

$\Downarrow$

1 $(A - I\lambda)^T\vec{x} = 0$ " " " "

$\Downarrow$

1 $(A^T - I\lambda)\vec{x} = 0$ " " " "

$\Downarrow$

1 λ is an eigen value of A^T

5. [12 points] Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

4/ **a.** Find the characteristic polynomial of A. List the eigenvalues and their multiplicities.

6/ **b.** Find the all the corresponding eigenvectors.

2/ **c.** If possible, diagonalize the matrix A. That is, find a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. (Note: You are not asked to find P^{-1}.)

a.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (-)\begin{vmatrix} 1 & 1 & -\lambda \\ 1 & -\lambda & 1 \\ -\lambda & 1 & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 1-\lambda & \lambda \\ 0 & \underbrace{-1-\lambda}_{-(1+\lambda)} & \lambda+1 \\ 0 & \lambda+1 & \underbrace{-\lambda^2+1}_{(1-\lambda)(1+\lambda)} \end{vmatrix} = (-1)(1+\lambda)(1+\lambda)\begin{vmatrix} 1 & 1 & -\lambda \\ 0 & -1 & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

4 points.

$$= (-1)(1+\lambda)^2\left[\overbrace{-1+\lambda \quad -1}^{\lambda-2}\right] = 0$$

$\lambda = -1$ ↑ mul 2 $\qquad \lambda = 2$ ↑ mul 1

3 points b. For eigenvalue $\lambda = -1$

$$[A - (-1)I \vdots 0] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 basic $\quad \underbrace{}_{x_2, x_3 \text{ free.}}$

$x_1 + x_2 + x_3 = 0$

$x_2 = t$ $\quad t, s \in \mathbb{R} - \{0\}$

$x_3 = s$

$x_1 = -t - s$

$x_2 = t$

$x_3 = s$ $\Rightarrow t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigenspace = S [-1] [-1]

3 points For eigenvalue $\lambda = 2$

$$[A - 2I : 0] = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free

$x_1 - x_3 = 0$ $\quad$ $x_1 = t$

$x_2 - x_3 = 0$ $\quad$ $x_2 = t$ $\quad$ $t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad$ $t \in \mathbb{R} - \{0\}$.

$x_3 = t$ $\quad$ $x_3 = t$

$\vec{v}_3$.

2 points

c). Yes

Let $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad$ $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

check if you want $A = PDP^{-1} \Rightarrow AP = PD$.

$$AP = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

Extra page