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Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1302 D – FINAL EXAM – April, 12th 2011

Instructor: Termeh Kousha

NAME____

I.D.#____

Duration: 3 hours

NO CALCULATORS. NO BOOKS. NO NOTES.

Instructions: This exam consists of 10 multiple choice questions and 8 Long answer questions on **16** pages. The marks for each question are as listed with the question itself. The total value of the exam is **55** points.
Place your answers to the multiple choice questions in the boxes below. You may use the back of pages for short and long answer questions.

Answers to Multiple Choice:

B	D	A	B	B
#1	#2	#3	#4	#5
A	B	A	B	A
#6	#7	#8	#9	#10

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Multiple Choice Questions, each is worth 2 marks, 20 marks overall.

1. For which values of h and k does the system

$$\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$$

have no solutions.

$$\begin{bmatrix} 1 & 3 & K \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & K \\ 0 & -12+h & -4K+8 \end{bmatrix}$$

(a) $h = 12,\ k = 2.$

(b) $h = 12,\ k \neq 2.$ ✓

→ the system is inconsistent if $h=12$ and $K \neq 2$

(c) $k \neq 2$ and $h \in \mathbb{R}$.

(d) For all $h, k \in \mathbb{R}$.

(e) For all values of k and h the system has solutions.

2. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A^T B$ is

(a) Not defined.

$$A^T B = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \\ 6 \end{bmatrix}$$

(b) $\begin{bmatrix} 11 \\ 7 \\ 4 \end{bmatrix}$.

(c) $\begin{bmatrix} 4 \\ 7 \\ 11 \end{bmatrix}$.

(d) $\begin{bmatrix} 15 \\ 2 \\ 6 \end{bmatrix}$. ✓

(e) $\begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

3. Let A and B be square matrices of size n. Which of the following statements are true?

1. $det(A) = 0$ if and only if the equation $Ax = 0$ has ONLY trivial solution.

2. A is invertible if and only if A has eigenvalue 0.

3. $det(AB) = det(BA)$. ✓

4. If $ColA = \mathbb{R}^n$, then $dimNulA = 0$. ✓

5. If λ is an eigenvalue of A, then λ is also an eigenvalue of A^T. ✓

(a) ✓ Statements 3,4,5 (b) Statements 1,2,4 (c) Statements 2,3,4,5

(d) Statements 1,4,5 (e) All of the statements are False!

4. Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -2a \\ b \\ -a \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Under which condition $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for $\mathbb{R}^3$?

$$= -\begin{bmatrix} 2a \\ -b \\ a \end{bmatrix}$$

(a) $a = 0,\ b \in \mathbb{R}$.

(b) ✓ $a \neq b$.

(c) $a \neq -b$.

(d) $a = -b$.

(e) $a = b$.

5. Let $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Compute A^3.

(a) $\begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix}$

(b) ✓ $\begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 8 & -8 \\ 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 16 \\ 0 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$$

$$A^3 = P D^3 P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 8 & -8 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 16 \\ 16 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix}$$

6. Suppose A, B and C are three 3×3 matrices with $detA = -1$, $detB = 2$, $detC = 3$. Find

$$\det(2A^5C^TB^TC^{-1})$$

$$= 2^3 det(A^5) det(C^T) det(B^T) det(C^{-1})$$

$$2^3 \ (-1) \ (3)(2) \left(\frac{1}{3}\right) = -16$$

(a) -16.

(b) 16.

(c) -4.

(d) 4.

(e) -36.

7. Let $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 2 & 4 \end{bmatrix}$, then

(a) B has 3 distinct eigenvalues and $detB$=16

(b) B has 2 distinct eigenvalues and $detB$=16

(c) B has 3 distinct eigenvalues and $detB$=9

(d) B has 1 distinct eigenvalue and $detB$=16

(e) B has 2 distinct eigenvalues and $detB$=9

8. Which of the following statements are true?

1. It is possible for a set of 4 vectors in $\mathbb{R}^3$ to be linearly independent.

2. Any basis of $\mathbb{R}^5$ must consist of 5 vectors.✓

3. It is possible for a set of 4 vectors to span $\mathbb{R}^3$✓

4. Any basis of a subspace contains the zero vector.

5. A subspace can have more than one basis.✓

(a) Statements 2,3,5 (b) Statements 1,2,4,5 (c) Statements 2,3,4,5

(d) Statements 1,4,5 (e) All of the statements are False!

9. Which sets are linearly independent?

$$A = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix} \right\}, \; B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}, \; D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(a) A and B.

(b) C and D.

(c) A and C.

(d) All of them.

(e) None of them.

10. Which of the following statements are true for an $n \times n$ matrix A ?

1. $\det A = \det A^T$

2. For every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a solution.

3. The equation $A\vec{x} = 0$ always has a solution.

4. $\dim(\text{Nul}A) \leq n$.

5. Rank$A$=n.

(a) Statements 1,3,4 (b) Statements 3,4,5, (c) Statements 1,2,4

(d) Statements 1,4,5 (e) All of the statements are True

Long Answer Questions:

1. [3 marks] Consider the production model $x = Cx + d$ for an economy with two sectors, where

$$C = \begin{bmatrix} 0 & .5 \\ .6 & 0.2 \end{bmatrix} \quad d = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

Determine the production level necessary to satisfy the final demand.

$$(I - C)\vec{x} = \vec{d}$$

$$\vec{x} = (I - C)^{-1}\vec{d}$$

$$(I - C)^{-1} = \left(\begin{bmatrix} 1 & -0.5 \\ -0.6 & 0.8 \end{bmatrix} \right)^{-1} = 2 \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 52 \end{bmatrix}$$

2. [3 marks] Let $z = 2 - 2i$, $w = -1 + i$ and $r = -2$. Calculate the following.

2 points

a. $\frac{\bar{z}}{w} = \dfrac{2 + 2i}{-1 + i} \times \dfrac{-1 - i}{-1 - i} = \dfrac{-4i}{2} = -2i$

$-2-2i$

1 point

b. $|rz| = |r||z| = 2\sqrt{4+4} = 2\sqrt{8} = 4\sqrt{2}$

3. [5 marks] Let $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$.

(a) Find the solution set of the homogeneous system $A\vec{x} = 0$ in parametric vector form. What is the geometric description?

(b) Describe the solution of $A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$ in the parametric vector form. Provide a geometric solution comparison with the solution set in part (a).

(a) $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \sim \cdots \left[\begin{array}{cccc} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \underbrace{0}_{} & 0 \end{array}\right]$

x_3 free.

2 points

$x_1 - 5x_3 = 0$

$x_2 + 2x_3 = 0$

$x_3 = t \quad t \in \mathbb{R}$

$t \overbrace{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}^{\vec{u}} \quad t \in \mathbb{R}$

Solution is $\text{span}\{\vec{u}\}$

0.5 point

Its a line passes through origin.

(b) $\left[\begin{array}{cccc} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right] \sim \cdots \left[\begin{array}{cccc} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \underbrace{0}_{} & 0 \end{array}\right]$

free.

2 points

$x_1 = -2 + 5t$

$x_2 = 1 - 2t$

$x_3 = t$

$t \underbrace{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}_{\vec{u}} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

The solution set is a line through the origin parallel to the line $\vec{u}$.

Question 3, continue

4. [7 marks] Let $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

a. Find the characteristic polynomial of A. List the eigenvalues and their multiplicities.

b. Find all the corresponding eigenvectors.

c. If possible, diagonalize the matrix A. That is, find a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. (Note: You are not asked to find P^{-1}.)

(a) 2 points

$$\begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)^2(4-\lambda) = 0$$

$\lambda = 4 \leftarrow$ mul 1.

$\lambda = 5 \leftarrow$ mul 2

2 points b) For $\underline{\lambda = 4}$

$$[A - \lambda I : 0] = \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 free

$x_1 = -\frac{1}{2}t$

$x_2 = t$

$x_3 = 0$

$t\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \rightarrow$ take $v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

2 points For $\lambda = 5$

$$[A - 5I : 0] \sim \cdots \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 x_3 free

$x_1 = -2x_3$

$x_2 = t$

$x_3 = s$

$\begin{bmatrix} -2s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

1 point

c). $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Question 4, continue

5. [3 marks] Is the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ invertible? If so, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{I} \qquad \underbrace{\qquad\qquad}_{A^{-1}}$$

6. [3 points] Find the determinant of $A = \begin{bmatrix} 2 & -4 & 6 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 3 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$.

$$|A| = (-1)\begin{vmatrix} 1 & 4 & 0 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -4 & 6 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 4 & 0 \\ 0 & -5 & 2 \\ 0 & -5 & 1 \end{vmatrix} + 2(2)\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underbrace{(-1)[-5+10]}_{-5} + 4\begin{vmatrix} 1 & -2 & 3 \\ 0 & 7 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$

$$\underbrace{4(\underbrace{-14+4}_{-10})}_{-40}$$

$$= -45$$

7. [6 marks] Consider the matrix $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}_{4\times 5.}$, whose reduced echelon form is $\begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $\sim \begin{bmatrix} 1 & 0 & -9 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

free

a. Find a bases for $ColA$. What is the $RankA$?

b. Find a bases for $NulA$. What is the dimension $NulA$?

c. State the equation relating the number of columns of a matrix, rank and the dimension of the null space.

a) Columns 1, 2, 4 have pivots.

2 points basis for COL A = $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix} \right\}$

Dim COL (A) = Rank A = 3

3 points b) ~~NUl A~~ From RREF: x_3, x_5 are free

$$x_1 - 9x_3 + 5x_5 = 0$$
$$x_2 + 2x_3 - 3x_5 = 0$$
$$x_4 + 2x_5 = 0$$

$$t \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$x_3 = t$

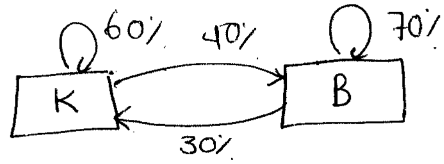
$x_5 = s$

Basis for Nul (A) = $\left\{ \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

Dim Nul A = 2.

c). Dim Nul A + Rank A = number of columns

$2 + 3 = 5$



8. [5 marks] There are 20 cockroaches living in a suite of a five star hotel. The cockroaches move between the kitchenette and bathroom. At the beginning, there are 10 cockroaches in the kitchenette and 10 in the bathroom. Every week 40% of the cockroaches in the kitchenette go to the bathroom and 30% of the cockroaches in the bathroom go to the kitchen.

$$\vec{x}_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \to \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

a. Determine the migration matrix associated with this model.

1 point

$$P = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

b. Compute the expected number of cockroaches in the bathroom after 2 weeks.

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 6+3 \\ 4+7 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix} \text{ after a week}$$

2 points

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 9 \\ 11 \end{bmatrix} = \begin{bmatrix} 5.4+3.3 \\ 3.6+7.7 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 11.3 \end{bmatrix} \text{ after two weeks}$$

$$\begin{bmatrix} 8.7/20 \\ 11.3/20 \end{bmatrix} \to$$

c. After the migration goes on for a long period of time, find the proportion of the cockroaches in the kitchenette and bathroom. (Hint: Find the steady-state vector of the migration matrix.)

2 points

$$P\vec{q} = \vec{q} \Rightarrow (P - I)\vec{q} = \vec{0}$$

$$\Rightarrow P - I = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{bmatrix}$$

$$[P - I : 0] \sim \begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 3/4\, x_2 \qquad \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 4 \end{bmatrix} \to \text{Probability vector } \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$$

x_2 free.

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Extra Page 2. Do NOT detach this page!