University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II
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Second Midterm Test (White Version) – Solutions
March 19, 2010

Surname ________________________ First Name ________________________

Student # ________________________ DGD (1–4) ________________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in **pen**, not pencil
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	6	4	5	4	4	5	28
Grade							

1.

(a) **[3 pts]** Let

$$A = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & -1 & 3 & 3 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

Compute the determinant of A.

Solution: We first expand along the second row.

$$\det A = 2 \cdot \begin{vmatrix} 3 & -1 & 0 \\ 1 & 3 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{vmatrix}$$

We then expand the first determinant along the third column and the second determinant along the third row.

$$\begin{aligned} \det A &= 2 \cdot (-3) \cdot \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} + 1 \cdot \left(2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \right) \\ &= -6(3 \cdot 1 - (-1) \cdot 2) + \big(2(2 \cdot 3 - (-1)(-1)) + (3(-1) - 2 \cdot 1)\big) \\ &= -6(5) + (2(5) + (-5)) \\ &= -30 + 5 \\ &= -25 \end{aligned}$$

(b) **[1 pt]** Find $\det B$, where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ -3 & 1 & 2 & 0 & 0 \\ 4 & -1 & 4 & -1 & 0 \\ 1 & 2 & 0 & 5 & -1 \end{bmatrix}.$$

Solution: Since the matrix is lower triangular, the determinant is simply the product of the entries on the diagonal. Thus

$$\det B = 1 \cdot (-1) \cdot 2 \cdot (-1) \cdot (-1) = -2.$$

(c) **[2 pts]** Suppose C is a 5×5 invertible matrix. Do you have enough information to compute $\det(2C^{-1}B^2C^T)$, where B is the matrix from Part (b)? If so, what is this determinant?

Solution: Yes, we have enough information.

$$\begin{aligned}\det(2C^{-1}B^2C^T) &= 2^5(\det C)^{-1}(\det B)^2(\det C^T)\\ &= 2^5 \cdot (-2)^2 \cdot (\det C)^{-1}(\det C) = 2^7 = 128.\end{aligned}$$

2. [**4 pts**] Suppose

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}.$$

Is M invertible? If so, find M^{-1}.

Solution: We form the superaugmented matrix and row reduce.

$$[\ M \mid I\ ] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 - R_1 \to R_4]{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 + R_2 \to R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_4 \to R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$\xrightarrow[R_2 + R_4 \to R_2]{R_3 - R_4 \to R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -7/2 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3/2 & -1/2 & 1 & -1/2 \\ 0 & 0 & 0 & 1 & -3/2 & 1/2 & 0 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/2 & 1/2 & -1 & 1/2 \\ 0 & 1 & 0 & 0 & -7/2 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3/2 & -1/2 & 1 & -1/2 \\ 0 & 0 & 0 & 1 & -3/2 & 1/2 & 0 & 1/2 \end{array}\right]$$

Since M is row equivalent to the identity matrix, it is invertible. The inverse is

$$M^{-1} = \begin{bmatrix} -1/2 & 1/2 & -1 & 1/2 \\ -7/2 & 3/2 & 0 & 1/2 \\ 3/2 & -1/2 & 1 & -1/2 \\ -3/2 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

3. An economy has two sectors: Agriculture and Manufacturing. In order to produce one unit of output, Agriculture requires 0.7 units from its own sector and 0.1 units from Manufacturing. On the other hand, Manufacturing requires 0.5 units from its own sector and 0.5 units from Agriculture to produce one unit of output.

(a) [**1 pt**] Write down the consumption matrix C for this economy.

Solution:

$$C = \begin{bmatrix} 0.7 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}.$$

(b) [**1 pt**] Determine the intermediate demands if Agriculture produces 100 units of output.

Solution: The unit consumption vector for Agriculture is

$$\begin{bmatrix} 0.7 \\ 0.1 \end{bmatrix}.$$

Therefore, the intermediate demand is

$$100 \begin{bmatrix} 0.7 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 70 \\ 10 \end{bmatrix}$$

or 70 units from Agriculture and 10 units from Manufacturing.

(c) [**3 pts**] Determine the production levels required to meet a final demand of 100 units from Agriculture and 150 units from Manufacturing.

Solution: We must solve the equation

$$\vec{x} = C\vec{x} + \vec{d} \quad \text{or} \quad (I - C)\vec{x} = \vec{d} \quad \text{or} \quad \vec{x} = (I - C)^{-1}\vec{d}.$$

We have

$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.5 \\ -0.1 & 0.5 \end{bmatrix}.$$

Using the formula for the inverse of a 2×2 matrix, we have

$$(I - C)^{-1} = \frac{1}{0.3 \cdot 0.5 - (-0.5)(-0.1)} \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.3 \end{bmatrix} = \frac{1}{0.1} \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.3 \end{bmatrix}$$

$$= 10 \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 1 & 3 \end{bmatrix}.$$

Therefore

$$\vec{x} = \begin{bmatrix} 5 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \end{bmatrix} = \begin{bmatrix} 500 + 750 \\ 100 + 450 \end{bmatrix} = \begin{bmatrix} 1250 \\ 550 \end{bmatrix}.$$

Therefore, in order to meet the given final demand, Agriculture must produce 1250 units and Manufacturing must produce 550 units.

4. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 3 & 1 & 0 \\ 1 & 0 & 3 & 0 & 8 & 8 & 0 \\ 1 & 0 & -1 & -2 & -2 & -3 & 0 \end{bmatrix}.$$

(a) **[2 pts]** Find a basis for Col A.

Solution: We first row reduce A to echelon form to find its pivot columns.

$$A \xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 & 5 & 7 & 0 \\ 0 & 0 & -2 & -1 & -5 & -4 & 0 \end{bmatrix} \xrightarrow{R_3+R_2\to R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

We see from the echelon form of A that the pivot columns are the first, third and sixth columns of A. These form a basis for Col A. So a basis for Col A is

$$\{(1,1,1),\ (1,3,-1),\ (1,8,-3)\}.$$

(b) **[1 pt]** What is rank A?

Solution: Since the rank of a matrix is the dimension of its column space (i.e. the number of elements in any basis of the column space), rank $A = 3$.

(c) **[1 pt]** What is the nullity of A? Remember that the nullity of A is dim Nul A.

Solution: Since A has 7 columns, by the Rank Theorem

$$\dim \operatorname{Nul} A = 7 - \operatorname{rank} A = 7 - 3 = 4.$$

5. Let

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) **[3 pts]** Find a basis for $\operatorname{Nul} B$.

Solution: Remember that the null space of B is the set of solutions to the homogeneous equation $B\vec{x} = \vec{0}$. Since the matrix B is already in reduced row echelon form, we can immediately write down the general solution:

$$\begin{aligned} x_1 &= -x_3 - 3x_5 \\ x_2 &= -2x_3 + x_5 - 4x_6 \\ x_4 &= -2x_5 - x_6 \\ x_7 &= 0 \\ x_3, x_5, x_6 & \quad \text{free} \end{aligned}$$

Switching to vector parametric notation, the solution set is

$$\vec{x} = \begin{bmatrix} -x_3 - 3x_5 \\ -2x_3 + x_5 - 4x_6 \\ x_3 \\ -2x_5 - x_6 \\ x_5 \\ x_6 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -4 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3, x_5, x_6 \in \mathbb{R}.$$

Therefore, a basis of $\operatorname{Nul} B$ is

$$\{(-1, -2, 1, 0, 0, 0, 0),\ (-3, 1, 0, -2, 1, 0, 0),\ (0, -4, 0, -1, 0, 1, 0)\}.$$

(b) **[1 pts]** What is $\operatorname{rank} B$?

Solution: The rank of B is the number of pivot positions in B. Therefore $\operatorname{rank} B = 4$.

6. [**5 pts**] Which of the following sets are subspaces of $\mathbb{R}^n$ for the given value of n? Remember to justify your answer.

(a) $A = \{(x, -x) \mid x \geq 0\}$, $n = 2$.

Solution: No, A is not a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication. For instance $(1, -1) \in A$ but $(-1)(1, -1) = (-1, 1) \notin A$.

(b) $B = \left\{ \begin{bmatrix} 3x + 2y - z \\ -x + y + z \\ 2x \\ -5y \\ 4 \end{bmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$, $n = 5$.

Solution: No, B is not a subspace of $\mathbb{R}^5$ since is does not contain the zero vector.

(c) $C = \left\{ \begin{bmatrix} 3x + 2y - z \\ -x + y + z \\ 2x \\ -5y \\ z \end{bmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$, $n = 5$.

Solution: Since

$$\begin{bmatrix} 3x + 2y - z \\ -x + y + z \\ 2x \\ -5y \\ z \end{bmatrix} = x \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 0 \\ -5 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

we see that

$$C = \text{Span}\{(3, -1, 2, 0, 0),\; (2, 1, 0, -5, 0)\,(-1, 1, 0, 0, 1)\}$$

and is therefore a subspace of $\mathbb{R}^5$.