University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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First Midterm Test (White Version) – Solutions

February 5, 2010

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1–4) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in **pen**, not pencil
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	3	5	4	4	4	25
Grade							

1. **[5 pts]** Is the following linear system consistent or inconsistent? If it is consistent, find the general solution.

$$\begin{aligned} x_1 + 2x_3 + 5x_4 + 7x_5 &= x_2 + 1 \\ -x_1 + 2x_2 &= x_2 + x_4 + x_5 - 3 \\ x_1 - x_2 + x_3 + 6x_4 + 5x_5 - 5 &= -x_1 + x_2 + 2x_4 \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & 5x_4 & + & 7x_5 & = & 1 \\ -x_1 & + & x_2 & & & - & x_4 & - & x_5 & = & -3 \\ 2x_1 & - & 2x_2 & + & x_3 & + & 4x_4 & + & 5x_5 & = & 5 \end{array}$$

Then write down the augmented matrix and row reduce to RREF:

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ -1 & 1 & 0 & -1 & -1 & -3 \\ 2 & -2 & 1 & 4 & 5 & 5 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 2 & 4 & 6 & -2 \\ 0 & 0 & -3 & -6 & -9 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{3}{2}R_2+R_3\to R_3} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 2 & 4 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 5 & 7 & 1 \\ 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

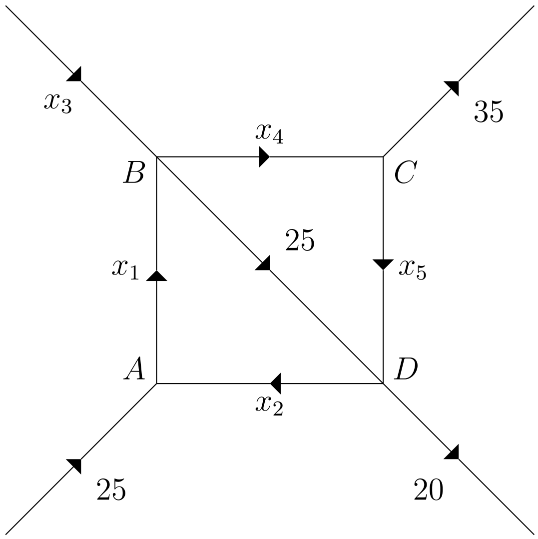
Since the rightmost column is not a pivot column, the system is consistent. Switching back to equation notation gives:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & & & + & x_4 & + & x_5 & = & 3 \\ & & & & x_3 & + & 2x_4 & + & 3x_5 & = & -1 \end{array}$$

Solving for the basic variables in terms of the free variables, we obtain the general solution:

$$\begin{aligned} x_1 &= x_2 - x_4 - x_5 + 3 \\ x_3 &= -2x_4 - 3x_5 - 1 \\ x_2, x_4, x_5 &\quad \text{free} \end{aligned}$$

2. [**3 pts**] Write down a system of equations describing the following traffic flow problem. The letters A through D label intersections and numerical values indicate flow in cars per minute. The arrows indicate the direction of flow (all roads are one-way). Include all relevant equations. You do **not** need to solve the system.



Solution: Setting the total flow in equal to the flow out at each intersection and for the overall system as a whole, we get the following linear system.

Intersection	**Flow in**		**Flow out**
A	$25 + x_2$	$=$	x_1
B	$x_1 + x_3$	$=$	$25 + x_4$
C	x_4	$=$	$35 + x_5$
D	$25 + x_5$	$=$	$20 + x_2$
Overall	$25 + x_3$	$=$	$20 + 35$

3. [**5 pts**] Find the general solution to the matrix-vector equation

$$\begin{bmatrix} 1 & -1 & 2 & 3 & -2 \\ 2 & -2 & 6 & 6 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Write your answer in **vector parametric form**.

Solution: We row reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -2 & 1 \\ 2 & -2 & 6 & 6 & 4 & 2 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -2 & 1 \\ 0 & 0 & 2 & 0 & 8 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 & 4 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & -10 & 1 \\ 0 & 0 & 1 & 0 & 4 & 0 \end{array}\right]$$

Switching to equation notation and solving for the basic variables in terms of the free variables, the general solution is:

$$\begin{aligned} x_1 &= x_2 - 3x_4 + 10x_5 + 1 \\ x_3 &= -4x_5 \\ x_2, x_4, x_5 &\quad \text{free} \end{aligned}$$

Switching to vector parametric form gives:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - 3x_4 + 10x_5 + 1 \\ x_2 \\ -4x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 10 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. [**4 pts**] Are the vectors $(1, 2, 3)$, $(-1, 1, 1)$ and $(1, 5, 7)$ linearly independent or linearly dependent?

Solution: The question is equivalent to asking if the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution. Therefore, we write down the corresponding augmented matrix (with the vectors in question as the columns of the coefficient matrix) and row reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ 3 & 1 & 7 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \xrightarrow{-\frac{4}{3}R_3+R_4\to R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the system is consistent (since the last column is not a pivot column) and the general solution has a free variable (since the third column is not a pivot column) and thus the vectors are linearly dependent.

5. [**4 pts**] Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ -2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -1 \\ 4 \\ 7 \\ 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}.$$

Is $\vec{b}$ in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?

Solution: We need to know if the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

has a solution. We write down the corresponding augmented matrix and row reduce:

$$\left[\begin{array}{rrr|r} 0 & 3 & -1 & 2 \\ 1 & -2 & 4 & 1 \\ 2 & -1 & 7 & 5 \\ 1 & -2 & 4 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & -2 & 4 & 1 \\ 0 & 3 & -1 & 2 \\ 2 & -1 & 7 & 5 \\ 1 & -2 & 4 & 1 \end{array}\right] \xrightarrow[-R_1+R_4 \to R_4]{-2R_1+R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & -2 & 4 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & -2 & 4 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last column is a pivot column, the system has no solution and thus the answer is **NO**, $\vec{b}$ cannot be written as a linear combination of $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$.

6. [**4 pts**] Suppose a linear system has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+5 & q+9 \end{array}\right]$$

for some real numbers p and q. For which values of p and q does the system have:

(a) No solution?
(b) Exactly one solution?
(c) Infinitely many solutions?

Your answers should include all possibilities for the values of p and q. Write your final answer in the spaces at the bottom of the page.

Solution: We row reduce the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & p+5 & q+9 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & p-4 & q+3 \end{array}\right]$$

This has a pivot in the last column (and hence no solution) when $p-4=0$ and $q+3 \neq 0$. In other words, when $p=4$ and $q \neq -3$.

The reduced matrix has pivots in both columns (and hence a unique solution) when $p-4 \neq 0$. In other words, when $p \neq 4$ (q can be any real number).

The system is consistent with a free variable (and hence has infinitely many solutions) when $p-4=0$ and $q+3=0$. In other words, when $p=4$ and $q=-3$.

Final Answer: (a) $p=4,\ q \neq -3$ (b) $p \neq 4,\ q \in \mathbb{R}$ (c) $p=4,\ q=-3$