University of Ottawa
Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II
Professor: Alistair Savage

Final Exam
April 2010

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 14 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 15 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) Each answer only question is worth two points. For the long answer questions, the number of points available for each question is indicated in square brackets.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–14	15	16	17	18	19	20	21	22	Total
Maximum	28	4	4	4	6	8	6	6	4	70
Grade										

Part A: Answer Only Questions

For Questions 1–14, only your final answer will be considered for marks. Each question is worth 2 points.

1. What is the determinant of the matrix

$$\begin{bmatrix} 4 & 7 & 10 & -4 & 0 \\ 0 & 1/2 & -6 & 7 & 8 \\ 0 & 0 & 3 & 4 & -5/2 \\ 0 & 0 & 0 & -2 & 7/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ?$$

Answer: _______________

2. Give the characteristic polynomial of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 5 & -6 & 3 & 0 \end{bmatrix}.$$

Answer: ______________________________

3. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and $\det A = 3$. If

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} a_{11} & a_{12} & a_{13} - 3a_{11} \\ a_{21} & a_{22} & a_{23} - 3a_{21} \\ a_{31} & a_{32} & a_{33} - 3a_{31} \end{bmatrix},$$

what are $\det B$ and $\det C$?

Answer: $\det B =$ __________ $\det C =$ __________

4. Is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ an eigenvector of the matrix $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$? If so, what is the corresponding eigenvalue?

Answer: _______________

5. Suppose A is an $n \times n$ matrix. Which of the following statements are equivalent to the statement "A is invertible"? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\det A \neq 0$.
(b) 0 is an eigenvalue of A.
(c) A has n distinct eigenvalues.
(d) $\operatorname{Col} A = \mathbb{R}^n$.
(e) A is diagonalizable.
(f) A^T is invertible.

Answer: _______________

6. Suppose that A, B and C are 3×3 matrices such that $\det A = 4$, $\det B = -1$, and $\det C = 3$. What is $\det\left(2C^3BA(C^T)^{-3}\right)$?

Answer: _______________

7. Which of the following are subspaces of $\mathbb{R}^n$ for the given value of n? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\{(x-y, 2y, 3x+y, 1) \mid x, y \in \mathbb{R}\}$, $n = 4$.
(b) $\{(2x+3y, x-z, y+2z, z) \mid x, y, z \in \mathbb{R}\}$, $n = 4$.
(c) $\{(x_1, x_2, x_3) \mid x_1 - 3x_2 + 4x_3 = 0\}$, $n = 3$.
(d) $\{(x_1, x_2, x_3) \mid x_1 - 3x_2 + 4x_3 = -2\}$, $n = 3$.
(e) $\{(1, 0, 0)\}$, $n = 3$.
(f) $\{(0, 0, 0, 0)\}$, $n = 4$.

Answer: _______________

8. Which of the following are subspaces of $\mathbb{R}^4$? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\operatorname{Nul} A$, where A is a 3×4 matrix.
(b) $\operatorname{Nul} A$, where A is a 4×3 matrix.
(c) $\operatorname{Col} A$, where A is a 3×4 matrix.
(d) $\operatorname{Col} A$, where A is a 4×3 matrix.
(e) The span of a set of linearly dependent vectors in $\mathbb{R}^4$.
(f) The span of a set of linearly independent vectors in $\mathbb{R}^4$.

Answer: _______________

9. If $z = 2 - i$ and $w = 1 + 3i$, compute $\overline{(\bar{z} + w)}$. Write your answer in the form $a + bi$ where a and b are real numbers.

Answer: _______________

10. Write the complex number $\frac{1+5i}{3-i}$ in the form $a + bi$ where a and b are real numbers.

Answer: _______________

11. Suppose A, B and C are invertible matrices. Solve the following equation for the matrix X (assume the sizes of the matrices are such that all of the operations are defined). In other words, express X in terms of A, B, C, their inverses, and their transposes.

$$A^T B X B^{-1} C^3 = A.$$

Answer: ______________________________

12. Which of the following imply that an $n \times n$ matrix A is **not** diagonalizable? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) A has fewer than n (distinct) eigenvalues.
(b) A is not invertible.
(c) A has an eigenvalue λ of multiplicity 2, whose corresponding eigenspace has dimension 1.
(d) There is no basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Answer: ______________

13. Which of the following statements are true for a 6×6 matrix A? More than one statement may be true. You should indicate **all** of the true statements.

(a) A and A^T have the same eigenvalues.
(b) A is invertible if and only if $A\vec{x} = \vec{b}$ has a solution for **every** $\vec{b}$ in $\mathbb{R}^6$.
(c) $\det(-A) = -\det A$.
(d) A is invertible if and only if every column of A is a pivot column.
(e) It is possible for the equation $A\vec{x} = \vec{0}$ to have no solutions.
(f) The dimension of the column space of A can be greater than 6.

Answer: ______________

14. Compute AB if

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 1 & 4 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

Answer:

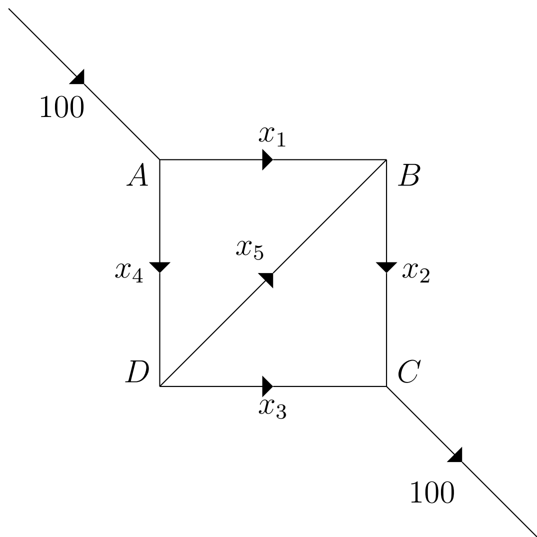
Part B: Long Answer Questions

For Questions 15–22, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

15. [**4 points**] Is the following linear system consistent or inconsistent? If it is consistent, find the general solution.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2x_4 + 6x_5 \\ -x_1 - x_3 + x_5 + 1 &= 2x_2 - 3x_4 \\ x_1 &= -2x_2 - 2x_3 + 10x_5 \end{aligned}$$

16. The traffic flow in a city is represented by the diagram below. The arrows indicate the direction of one-way traffic.



(a) **[2 points]** Write down a linear system describing this traffic flow. Do not perform any calculations at this stage.

(b) [**1 point**] The reduced echelon form of the augmented matrix in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Give the general solution of the system (ignore the constraints on the variables for now).

(c) [**1 point**] If road DB is closed due to construction and if the flow on road AD is $x_4 = 10$, what is the traffic flow on road BC?

17. [**4 points**] For which values of the parameters p and q are the vectors $(1, 0, 1, 0)$, $(2, -1, 4, 2)$, $(3, 0, p + 2, 0)$ and $(0, 1, 5, 3q - 2)$ linearly independent ?

18. I tend to be rather moody at times. If I am in a good mood today, there is an 80% chance I will still be in a good mood tomorrow and 20% chance that I will be grumpy tomorrow. But if I am grumpy today, there is only a 60% chance that my mood will be good tomorrow.

(a) [**1 point**] Determine the migration (transition) matrix for this situation.

(b) [**2 points**] If I am in a good mood on Monday, what is the probability that I will be in a good mood on the coming Wednesday?

(c) [**3 points**] In the long term, what percentage/fraction of the time am I in a good mood? *Hint:* Find the steady-state vector of the migration matrix and justify why it describes the long term behaviour.

19. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

(a) [**2 points**] Write down the characteristic equation of A. (Note: You are not asked to find the eigenvalues at this stage.)

(b) [**1 point**] Use the characteristic equation from part (a) to find the eigenvalues of A. For each eigenvalue, give its multiplicity.

(c) [**2 points**] Find a basis of the eigenspace of eigenvalue 2.

(d) [**2 points**] Find a basis of the eigenspace of eigenvalue 4.

(e) [**1 point**] Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (Note: You are not asked to find P^{-1}.)

20. Let M be the following matrix:

$$M = \begin{bmatrix} -1 & 2 & -3 \\ 3 & -2 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) [**4 points**] Is M invertible? If yes, find M^{-1}.

(b) [**2 points**] Solve for $\vec{x}$ in the equation $M\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

21. Let $A = \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 2 & -4 & 4 & -2 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

(a) [**3 points**] Find a basis for $\operatorname{Col} A$.

(b) [**3 points**] Find a basis for $\operatorname{Nul} A$.

22. An economy consists of two sectors: Agriculture and Industry. To produce one unit of output, Agriculture requires 0.2 units of its own products and 0.3 units from Industry. Moreover, Industry requires 0.6 units from its own products and 0.4 units from Agriculture to produce one unit of output.

(a) [**1 point**] Write down the consumption matrix C.

(b) [**3 points**] Calculate the production levels required to meet a final demand of 10 units from Agriculture and 6 units from Industry.

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