University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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Final Exam – Solutions

April 2010

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) This exam consists of two parts. Questions 1 through 14 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 15 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

(c) Each answer only question is worth two points. For the long answer questions, the number of points available for each question is indicated in square brackets.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–14	15	16	17	18	19	20	21	22	Total
Maximum	28	4	4	4	6	8	6	6	4	70
Grade										

Part A: Answer Only Questions

For Questions 1–14, only your final answer will be considered for marks. Each question is worth 2 points.

1. What is the determinant of the matrix

$$\begin{bmatrix} 4 & 7 & 10 & -4 & 0 \\ 0 & 1/2 & -6 & 7 & 8 \\ 0 & 0 & 3 & 4 & -5/2 \\ 0 & 0 & 0 & -2 & 7/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ?$$

Answer: -12

2. Give the characteristic polynomial of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 5 & -6 & 3 & 0 \end{bmatrix}.$$

Answer: $\lambda(\lambda-1)(\lambda-2)^2$

3. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

and $\det A = 3$. If

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} a_{11} & a_{12} & a_{13}-3a_{11} \\ a_{21} & a_{22} & a_{23}-3a_{21} \\ a_{31} & a_{32} & a_{33}-3a_{31} \end{bmatrix},$$

what are $\det B$ and $\det C$?

Answer: $\det B = -3$, $\det C = 3$

4. Is $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ an eigenvector of the matrix $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$? If so, what is the corresponding eigenvalue?

Answer: Since

$$\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1-i \\ -i \end{bmatrix} = -i \begin{bmatrix} 1+i \\ 1 \end{bmatrix},$$

the answer is YES and the eigenvalue is $-i$.

5. Suppose A is an $n \times n$ matrix. Which of the following statements are equivalent to the statement "A is invertible"? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\det A \neq 0$.
(b) 0 is an eigenvalue of A.
(c) A has n distinct eigenvalues.
(d) $\operatorname{Col} A = \mathbb{R}^n$.
(e) A is diagonalizable.
(f) A^T is invertible.

Answer: (a), (d), (f)

6. Suppose that A, B and C are 3×3 matrices such that $\det A = 4$, $\det B = -1$, and $\det C = 3$. What is $\det\left(2C^3BA(C^T)^{-3}\right)$?

Answer: -32

7. Which of the following are subspaces of $\mathbb{R}^n$ for the given value of n? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\{(x - y, 2y, 3x + y, 1) \mid x, y \in \mathbb{R}\}$, $n = 4$.
(b) $\{(2x + 3y, x - z, y + 2z, z) \mid x, y, z \in \mathbb{R}\}$, $n = 4$.
(c) $\{(x_1, x_2, x_3) \mid x_1 - 3x_2 + 4x_3 = 0\}$, $n = 3$.
(d) $\{(x_1, x_2, x_3) \mid x_1 - 3x_2 + 4x_3 = -2\}$, $n = 3$.
(e) $\{(1, 0, 0)\}$, $n = 3$.
(f) $\{(0, 0, 0, 0)\}$, $n = 4$.

Answer: (b),(c), (f)

8. Which of the following are subspaces of $\mathbb{R}^4$? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) $\operatorname{Nul} A$, where A is a 3×4 matrix.
(b) $\operatorname{Nul} A$, where A is a 4×3 matrix.
(c) $\operatorname{Col} A$, where A is a 3×4 matrix.
(d) $\operatorname{Col} A$, where A is a 4×3 matrix.
(e) The span of a set of linearly dependent vectors in $\mathbb{R}^4$.
(f) The span of a set of linearly independent vectors in $\mathbb{R}^4$.

Answer: (a), (d), (e), (f)

9. If $z = 2 - i$ and $w = 1 + 3i$, compute $\overline{(\bar{z} + w)}$. Write your answer in the form $a + bi$ where a and b are real numbers.

Answer: $3 - 4i$

10. Write the complex number $\frac{1+5i}{3-i}$ in the form $a + bi$ where a and b are real numbers.

Answer: $\frac{-1}{5} + \frac{8}{5}i$

11. Suppose A, B and C are invertible matrices. Solve the following equation for the matrix X (assume the sizes of the matrices are such that all of the operations are defined). In other words, express X in terms of A, B, C, their inverses, and their transposes.

$$A^T B X B^{-1} C^3 = A.$$

Answer: $X = B^{-1}(A^T)^{-1} A C^{-3} B$

12. Which of the following imply that an $n \times n$ matrix A is **not** diagonalizable? More than one answer may be correct. You should indicate **all** of the correct answers.

(a) A has fewer than n (distinct) eigenvalues.
(b) A is not invertible.
(c) A has an eigenvalue λ of multiplicity 2, whose corresponding eigenspace has dimension 1.
(d) There is no basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Answer: (c), (d)

13. Which of the following statements are true for a 6×6 matrix A? More than one statement may be true. You should indicate **all** of the true statements.

(a) A and A^T have the same eigenvalues.
(b) A is invertible if and only if $A\vec{x} = \vec{b}$ has a solution for **every** $\vec{b}$ in $\mathbb{R}^6$.
(c) $\det(-A) = -\det A$.
(d) A is invertible if and only if every column of A is a pivot column.
(e) It is possible for the equation $A\vec{x} = \vec{0}$ to have no solutions.
(f) The dimension of the column space of A can be greater than 6.

Answer: (a), (b), (d)

14. Compute AB if

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 1 & 4 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}.$$

Answer: $\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 1 \\ 1 & 8 & -13 \end{bmatrix}$

Part B: Long Answer Questions

For Questions 15–22, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

15. [**4 points**] Is the following linear system consistent or inconsistent? If it is consistent, find the general solution.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2x_4 + 6x_5 \\ -x_1 - x_3 + x_5 + 1 &= 2x_2 - 3x_4 \\ x_1 &= -2x_2 - 2x_3 + 10x_5 \end{aligned}$$

Solution: We first write the system in standard form:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & - & 2x_4 & - & 6x_5 & = & 0 \\ -x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & + & x_5 & = & -1 \\ x_1 & + & 2x_2 & + & 2x_3 & & & - & 10x_5 & = & 0 \end{array}$$

Then write down the augmented matrix and row reduce to RREF:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ -1 & -2 & -1 & 3 & 1 & -1 \\ 1 & 2 & 2 & 0 & -10 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_1 + R_2 \to R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ 0 & 0 & 1 & 1 & -5 & -1 \\ 0 & 0 & 0 & 2 & -4 & 0 \end{array}\right]$$

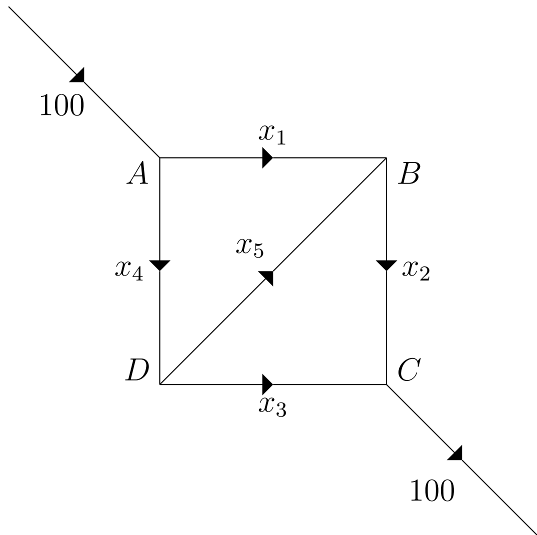
$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & -2 & -6 & 0 \\ 0 & 0 & 1 & 1 & -5 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_1 + 2R_3 \to R_1]{R_2 - R_3 \to R_2} \left[\begin{array}{rrrrr|r} 1 & 2 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

Since the last column is not a pivot column, the system is consistent. Solving for the basic variables in terms of the free variables, we obtain the general solution:

$$\begin{aligned} x_1 &= 2 - 2x_2 + 4x_5 \\ x_3 &= -1 + 3x_5 \\ x_4 &= 2x_5 \\ x_2, x_5 & \quad \text{free} \end{aligned}$$

16. The traffic flow in a city is represented by the diagram below. The arrows indicate the direction of one-way traffic.



(a) [**2 points**] Write down a linear system describing this traffic flow. Do not perform any calculations at this stage.

Solution:

Intersection	Flow in		Flow out
A	100	=	$x_1 + x_4$
B	$x_1 + x_5$	=	x_2
C	$x_2 + x_3$	=	100
D	x_4	=	$x_3 + x_5$
Overall	100	=	100

(b) [**1 point**] The reduced echelon form of the augmented matrix in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 100 \\ 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Give the general solution of the system (ignore the constraints on the variables for now).

Solution: Since the fourth and the fifth columns do not contain pivot elements, x_4 and x_5 are free variables. The general solution is:

$$\begin{aligned} x_1 &= 100 - x_4 \\ x_2 &= 100 - x_4 + x_5 \\ x_3 &= x_4 - x_5 \\ x_4, x_5 & \quad \text{free} \end{aligned}$$

(c) [**1 point**] If road DB is closed due to construction and if the flow on road AD is $x_4 = 10$, what is the traffic flow on road BC?

Solution: The two conditions give $x_4 = 10$ and $x_5 = 0$. Substituting these values into the general solution of the system, we obtain $x_2 = 100 - 10 = 90$.

17. [**4 points**] For which values of the parameters p and q are the vectors $(1,0,1,0)$, $(2,-1,4,2)$, $(3,0,p+2,0)$ and $(0,1,5,3q-2)$ linearly independent ?

Solution: We check if the corresponding linear homogeneous system has non trivial solutions:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 4 & p+2 & 5 & 0 \\ 0 & 2 & 0 & 3q-2 & 0 \end{array}\right] \xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & p-1 & 5 & 0 \\ 0 & 2 & 0 & 3q-2 & 0 \end{array}\right]$$

$$\xrightarrow[2R_2+R_4\to R_4]{2R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & p-1 & 7 & 0 \\ 0 & 0 & 0 & 3q & 0 \end{array}\right].$$

This system has no nontrivial solutions exactly when it has a pivot in each column. This occurs if and only if $p \neq 1$ and $q \neq 0$. Hence, the four vectors are linearly independent if and only if $p \neq 1$ and $q \neq 0$.

18. I tend to be rather moody at times. If I am in a good mood today, there is an 80% chance I will still be in a good mood tomorrow and 20% chance that I will be grumpy tomorrow. But if I am grumpy today, there is only a 60% chance that my mood will be good tomorrow.

(a) [**1 point**] Determine the migration (transition) matrix for this situation.

Solution: The transition matrix is

$$M = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}.$$

(b) [**2 points**] If I am in a good mood on Monday, what is the probability that I will be in a good mood on the coming Wednesday?

Solution: If I am in a good mood on Monday, then the corresponding probability vector is $\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For Tuesday, we have $\vec{p}_1 = M\vec{p}_0 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$.

On Wednesday, $\vec{p}_2 = M\vec{p}_1 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$. Therefore, there is a 76% chance that I will be in a good mood on Wednesday.

(c) [**3 points**] In the long term, what percentage/fraction of the time am I in a good mood? *Hint:* Find the steady-state vector of the migration matrix and justify why it describes the long term behaviour.

Solution: Since the matrix M is regular stochastic, the long term behaviour is given by the steady-state vector. To find the steady-state vector, solve $(M - I)\vec{x} = \vec{0}$.

$$M - I = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}.$$

To find x_2 solve $3x_2 + x_2 = 1$ to get $x_2 = 1/4$. Therefore the steady-state vector is

$$\begin{bmatrix} \frac{3}{4} \\ \\ \frac{1}{4} \end{bmatrix}.$$

The percentage of time that I am in a good mood is 75% or the fraction of time that I am in a good mood is $\frac{3}{4}$.

19. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

(a) [**2 points**] Write down the characteristic equation of A. (Note: You are not asked to find the eigenvalues at this stage.)

Solution: Expanding along the first row, we have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} \\ &= (4-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} \\ &= (4-\lambda)((3-\lambda)^2 - 1). \end{aligned}$$

Thus, the characteristic equation is $(4-\lambda)((3-\lambda)^2-1) = 0$ or $-(\lambda-4)^2(\lambda-2) = 0$.

(b) [**1 point**] Use the characteristic equation from part (a) to find the eigenvalues of A. For each eigenvalue, give its multiplicity.

Solution: The eigenvalues are 2 (with multiplicity 1) and 4 (with multiplicity 2).

(c) [**2 points**] Find a basis of the eigenspace of eigenvalue 2.

Solution: We row reduce the matrix $A - 2I$ to find the general solution to the homogenous system $(A - 2I)\vec{x} = \vec{0}$:

$$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

A basis of the eigenspace is thus $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) [**2 points**] Find a basis of the eigenspace of eigenvalue 4.

Solution: We row reduce the matrix $A - 4I$ to find the general solution to the homogenous system $(A - 4I)\vec{x} = \vec{0}$:

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

A basis of the eigenspace is thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(e) [**1 point**] Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (Note: You are not asked to find P^{-1}.)

Solution:

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

20. Let M be the following matrix:

$$M = \begin{bmatrix} -1 & 2 & -3 \\ 3 & -2 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) [**4 points**] Is M invertible? If yes, find M^{-1}.

Solution: We form the superaugmented matrix and row reduce:

$$[\ M \mid I\ ] = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 3 & -2 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1 \to R_3]{R_2+3R_1 \to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 4 & -4 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2 \to R_3} \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 4 & -4 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right] \xrightarrow[R_1+3R_3 \to R_1]{R_2+4R_3 \to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & -1/2 & -3/2 & 3 \\ 0 & 4 & 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & -1/2 & -3/2 & 3 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1/4 & -1/4 & 1 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

Since M is row equivalent to the identity matrix, it is invertible. The inverse is

$$M^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1/4 & -1/4 & 1 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

(b) [**2 points**] Solve for $\vec{x}$ in the equation $M\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution:

(c) Multiplying the equation on the left by M^{-1}, we find that

$$\vec{x} = M^{-1}\vec{u} = \begin{bmatrix} 1 & 1 & -1 \\ 1/4 & -1/4 & 1 \\ -1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

21. Let $A = \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 2 & -4 & 4 & -2 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

(a) **[3 points]** Find a basis for Col A.

Solution:

$$\begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 2 & -4 & 4 & -2 \\ -1 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third columns are pivot columns. It follows that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

is a basis for Col A.

(b) **[3 points]** Find a basis for Nul A.

Solution: We have to find the RREF of the augmented matrix of the homogeneous system $A\vec{x} = \vec{0}$, which is $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -\frac{1}{2} & 0 \\ 2 & -4 & 4 & -2 & 0 \\ -1 & 2 & 0 & 0 & 0 \end{array}\right]$.

From part (a) it follows that the RREF of this matrix is $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

which is the augemented matrix of the linear system $\begin{cases} x_1 - 2x_2 = 0, \\ x_3 - \frac{1}{2}x_4 = 0. \end{cases}$

The variables x_1, x_3 are basic and x_2, x_4 are free. The general solution, in vector parametric notation, is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ \frac{1}{2}x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

22. An economy consists of two sectors: Agriculture and Industry. To produce one unit of output, Agriculture requires 0.2 units of its own products and 0.3 units from Industry. Moreover, Industry requires 0.6 units from its own products and 0.4 units from Agriculture to produce one unit of output.

(a) [**1 point**] Write down the consumption matrix C.

Solution:
$C = \begin{bmatrix} .2 & .4 \\ .3 & .6 \end{bmatrix}$

(b) [**3 points**] Calculate the production levels required to meet a final demand of 10 units from Agriculture and 6 units from Industry.

Solution: We must solve the Leontief Input-Output Model equation

$$(I - C)\vec{x} = \vec{d}.$$

Since

$$I - C = \begin{bmatrix} 1 - .2 & -.4 \\ -.3 & 1 - .6 \end{bmatrix} = \begin{bmatrix} .8 & -.4 \\ -.3 & .4 \end{bmatrix},$$

we have

$$(I - C)^{-1} = \frac{1}{.20} \begin{bmatrix} .4 & .4 \\ .3 & .8 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} & 4 \end{bmatrix}.$$

Therefore the production level required is given by

$$\mathbf{x} = (I - C)^{-1} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 32 \\ 39 \end{bmatrix}$$

That is, Agriculture must produce 32 units and Manufacturing must produce 39 units.