University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

Instructor: Alistair Savage

Second Midterm Exam

Surname ____________________ First Name ____________________

Student # ____________________

Instructions:

(1) You have 80 minutes to complete this exam.
(2) The number of points available for each question is indicated in square brackets.
(3) You must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(4) All work to be considered for grading should be written in the space provided. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**.
(5) Write your student number at the top of each page in the space provided.
(6) No notes, books, calculators or scrap papers are allowed.
(7) The final page of the exam may be used for scrap work.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	6	6	7	3	4	30
Score							

1. [**4pt.**] Let $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$, and $\overrightarrow{v_3} = \begin{bmatrix} -3 \\ 15 \\ h \end{bmatrix}$. Find the value(s) of h for which the vectors are linearly *dependent*.

Solution: Let $A = [\overrightarrow{v_1} \quad \overrightarrow{v_2} \quad \overrightarrow{v_3}]$.

Method 1: The columns of A are dependent if $A\overrightarrow{x} = \overrightarrow{0}$ has infinitely many solutions or has at least one free variable. Write and reduce the augmented matrix of the equation:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ -1 & 5 & 15 & 0 \\ 0 & 3 & h & 0 \end{array}\right] \xrightarrow{R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 6 & 12 & 0 \\ 0 & 3 & h & 0 \end{array}\right] \xrightarrow{\frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & h & 0 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & h-6 & 0 \end{array}\right],$$

the first and the second columns have pivots so they correspond to basic variables. The third column corresponds to a free variable if

$$h - 6 = 0 \Rightarrow h = 6.$$

Method 2: Since A is a square matrix, its columns are dependent if the matrix is not invertible or its determinant is zero:

$$\begin{vmatrix} 1 & 1 & -3 \\ -1 & 5 & 15 \\ 0 & 3 & h \end{vmatrix} = (-1)^2(1)(5h-45) + (-1)^3(-1)(h+9) = 5h - 45 + h + 9 = 6h - 36 = 0$$

$$\Rightarrow h = 6.$$

2. The following matrices are given

$$A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}.$$

(a) **[4pt.]** Determine the inverse of C.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2\to R_2]{-2R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3\to R_3]{-R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Hence, the inverse of C is

$$C^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

(b) **[2pt.]** If possible, solve the equation $CX = A$ for X.

Solution: Multiply the equation from left by C^{-1} to get

$$X = C^{-1}A.$$

Hence,

$$X = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 3 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 3 & -1 \\ -4 & 4 \end{bmatrix}.$$

3. An economy has two sectors: Electricity and Services. For each unit of output, Electricity requires 0.5 units from its own sector and 0.4 units from Services. Meanwhile, Services requires 0.5 units from Electricity and 0.2 units from its own sector to produce one unit of Services.

(a) **[1pt.]** Determine the consumption matrix C.

Solution: $C = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.2 \end{bmatrix}$.

(b) **[1pt.]** State the Leontief input-output equation relating C to the production vector $\overrightarrow{x}$ and final demand vector $\overrightarrow{d}$.

Solution: $\overrightarrow{x} = C\overrightarrow{x} + \overrightarrow{d}$.

(c) **[4pt.]** Use an inverse matrix to determine the production vector necessary to satisfy a final demand of 1000 units of Electricity and 2000 units of Services, i.e. $\overrightarrow{d} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$.

Solution: We need to solve $(I - C)\overrightarrow{x} = \overrightarrow{d}$.

$I - C = \begin{bmatrix} 0.5 & -0.5 \\ -0.4 & 0.8 \end{bmatrix} \Rightarrow \det(I - C) = (0.5)(0.8) - (-0.5)(-0.4) = 0.2,$

$$(I - C)^{-1} = \frac{1}{0.2}\begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} = 5\begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 0.5 \end{bmatrix} = \begin{bmatrix} 4 & 2.5 \\ 2 & 2.5 \end{bmatrix}$$

Therefore,

$$\overrightarrow{x} = (I - C)^{-1}\overrightarrow{d} = \begin{bmatrix} 4 & 2.5 \\ 2 & 2.5 \end{bmatrix}\begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 9000 \\ 7000 \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 4 & 4 & -4 & 0 \\ 2 & 5 & -4 & -2 \end{bmatrix}$.

(a) **[3pt.]** Find a basis for Col A.

Solution: The pivot columns of A form a basis for the column space. Reduce A to find the pivot columns:

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 4 & 4 & -4 & 0 \\ 2 & 5 & -4 & -2 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{-4R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 4 & -16 & -16 \\ 0 & 5 & -10 & -10 \end{bmatrix} \xrightarrow{\frac{1}{4}R_2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -4 & -4 \\ 0 & 5 & -10 & -10 \end{bmatrix}$$

$$\xrightarrow{-5R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 10 & 10 \end{bmatrix} \xrightarrow{\frac{1}{10}R_3} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-3R_3+R_1\to R_1]{4R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The first, the second and the third columns are pivot columns, hence a basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix} \right\}$.

(b) **[2pt.]** Provide a basis for Nul A.

Solution: Write the general solution of $A\overrightarrow{x} = \overrightarrow{0}$, in vector parametric form. The augmented matrix of the equation is

$$[A|\overrightarrow{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right],$$

The general solution is

$$\begin{bmatrix} -x_4 \\ 0 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

A basis for the null space is $\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(c) **[2pt.]** Determine the rank of A and the dimension of the null space of A.

Solution: Count the number of elements in each basis to find the dimension of the spaces:

$$\text{rank}(A) = \dim \text{Col}(A) = 3, \text{ and } \dim \text{Nul}(A) = 1.$$

5. **[3pt.]** Calculate the following determinant:

$$\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 2 & 1 & 1 & 6 \end{vmatrix}.$$

Remember to show your work.

Solution: determinant$=(-1)^{1+1}(2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 6 \end{vmatrix} + 0 + (-1)^{1+3}(2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 6 \end{vmatrix} + 0$

$$= 2\left[(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} + 0 + (-1)^{3+1}(1)\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right] + 2\left[0 + 0 + (-1)^{2+3}(2)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 2[(18-2)+(2-3)] + 2[(-1)(2)(1-2)] = 34.$$

6. Find out which of the following subsets of $\mathbb{R}^4$ are subspaces. Justify your answer.

(a) **[2pt.]** $H = \left\{ \begin{bmatrix} x \\ y \\ u \\ z \end{bmatrix} \;\middle|\; x = y \right\}$.

Solution: Any member of H is of the form:

$$\begin{bmatrix} x \\ x \\ u \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

Thus $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and is a subspace.

(b) **[2pt.]** $K = \left\{ \begin{bmatrix} x \\ y \\ u \\ z \end{bmatrix} \;\middle|\; x \geq 0 \right\}$.

Solution: K is not a subspace since it is not closed under scalar multiplication. For example,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in K, \text{ but } -5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ is not in } K.$$

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