University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II
Instructor: Alistair Savage

Final Exam – April 23, 2009

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(1) You have 180 minutes to complete this exam.
(2) This exam consists of two parts. Questions 1 through 10 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 11 through 20 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(3) Each answer only question is worth two points. For the long answer questions, the number of points available for each question is indicated in square brackets.
(4) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(5) Write your student number at the top of each page in the space provided.
(6) No notes, books, calculators or scrap paper are allowed.
(7) The final page of the exam may be used for scrap work.
(8) Good luck!

Please do not write in the table below.

Question	1–10	11	12	13	14	15	16	17	18	19	20	Total
Maximum	20	4	4	4	4	4	5	8	6	5	6	70
Score												

Part A: Answer Only Questions

For Questions 1–10, only your final answer will be considered for marks. Each question is worth 2 points.

1. If

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$

what is AB?

Answer: _______________

2. If $z = 1 + 2i$ and $w = 3 - 2i$, compute $\bar{z}w$. Write your answer in the form $a + bi$ where a and b are real numbers.

Answer: _______________

3. Write the complex number $\frac{2-3i}{1+2i}$ in the form $a + bi$ where a and b are real numbers.

Answer: _______________

4. If $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 3$, what is $\det \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$?

Answer: _______________

5. What are the eigenvalues of the matrix

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 10 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & -7 & 6 & 8 & 0 \\ 2 & -4 & 6 & 3 & 10 \end{bmatrix}?$$

Answer: _______________

6. Which of the following sets are subspaces of $\mathbb{R}^n$ for the given value of n?

(a) $\{(0,0)\}$, $n = 3$
(b) $\text{Span}\{(1,0,-1),(2,2,0)\}$, $n = 3$
(c) $\{(x,x,y,-y) \mid x,y \in \mathbb{R}\}$, $n = 4$
(d) $\{(w,x,y,z) \mid w,x,y,z \in \mathbb{R},\ 2x+3y-z=1\}$, $n = 4$
(e) $\{(w,x,y,z) \mid w,x,y,z \in \mathbb{R},\ 2x+3y+z=0\}$, $n = 4$

Answer: _______________

7. Suppose P, Q and R are 3×3 matrices with

$$\det P = -1, \quad \det Q = 3, \quad \det R = 5.$$

What is $\det(2P^5Q^TRQ^{-1}R^T)$?

Answer: _______________

8. Which of the following statements are true for an $n \times n$ matrix M? Note that more than one statement may be correct (you should indicate **all** of the correct statements).

(a) $\det M = \det M^T$.
(b) M and M^T can have different eigenvalues.
(c) For every $\vec{b} \in \mathbb{R}^n$, the equation $M\vec{x} = \vec{b}$ has a solution.
(d) The equation $M\vec{x} = \vec{0}$ always has a solution.
(e) M is invertible if and only if zero is not an eigenvalue of M.
(f) M has at most n eigenvalues.

Answer: ______________

9. Which of the following statements are true? Note that more than one statement may be correct (you should indicate **all** of the correct statements).

(a) It is possible for a set of 5 vectors in $\mathbb{R}^4$ to be linearly independent.
(b) Any basis of $\mathbb{R}^5$ must consist of 5 vectors.
(c) Any subspace contains the zero vector.
(d) Any basis of a subspace contains the zero vector.
(e) It is possible for a set of 4 vectors to span $\mathbb{R}^3$.
(f) A subspace can have more than one basis.

Answer: ______________

10. What is the rank of

$$\begin{bmatrix} 0 & 8 & 0 & 9 & 10 & 5 & 6 & 3 \\ 0 & 0 & -1 & 0 & 8 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & -2 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}?$$

Answer: ______________

Part B: Long Answer Questions

For Questions 11–20, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

11. [**4**] Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix}$.

12. [**4**] Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \end{bmatrix}$ linearly dependent? If so, find a linear dependence relation.

13. [**4**] Is the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ invertible? If so, find its inverse A^{-1}.

14. [**4**] Find the value(s) of t for which $\begin{bmatrix} 3 \\ 3 \\ 7 \\ t \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -5 \\ 0 \\ 2 \end{bmatrix}$.

15. [**4**] Find a basis for the space of solutions of the following system:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & + & 4x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 7x_3 & + & 10x_4 & = & 0 \end{array}$$

16. [**5**] Consider an economy with two sectors: work and leisure. In order to produce one unit, the work sector uses $\frac{1}{3}$ of a unit from the work sector and $\frac{1}{6}$ of unit from the leisure sector. To produce one unit, the leisure section uses $\frac{1}{6}$ of a unit of each of the work and leisure sectors.

(a) [**1**] Write the consumption matrix C corresponding to this economy.

(b) [**1**] State the Leontief input-output equation relating the production vector $\vec{x}$ and the final demand vector $\vec{d}$.

(c) [**3**] Find the production $\vec{x}$ needed to satisfy a final demand of $\vec{d} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$.

17. [**8**] Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) [**3**] Use the characteristic equation to find the eigenvalues of A.

(b) [**2**] Find a basis of the eigenspace of eigenvalue -2.

(c) **[2]** Find a basis of the eigenspace of eigenvalue 1.

(d) **[1]** Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

18. [**6**] Two telephone companies *OutOfOrder* and *NoService* are competing. A statistical study has shown that in each **6 month** period, 60% of the clients of OutOfOrder stay with the company while 40% of them which to NoService. During the same period, 20% of the clients of NoService switch to OutOfOrder, while 80% stay with NoService

On January 1, 2009, OutOfOrder had 50 thousand clients and NoService had 25 thousand clients.

(a) [**1**] Write the migration matrix M.

(b) [**1**] How many clients will each company have on July 1, 2009?

(c) [**1**] How many clients will each company have on January 1, 2010?

(d) **[3]** Assuming the migration matrix M stays constant in the long term, find the market share (i.e. percentage of the total customers) of each company in the long term. That is, find the equilibrium market shares.

19. [**5**] Consider the matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$, whose reduced echelon form is $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

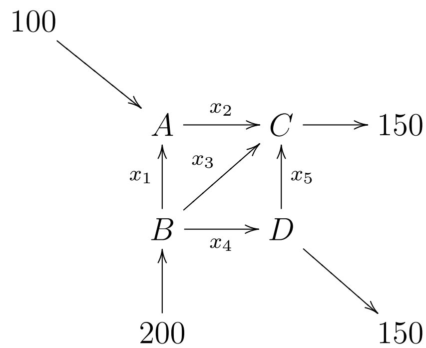
(a) [**2**] Find a basis of $\operatorname{Nul} A$.

(b) [**1**] Find the dimension of $\operatorname{Nul} A$.

(c) [**1**] Find a basis of $\operatorname{Col} A$.

(d) [**1**] Find the dimension of $\operatorname{Col} A$.

20. **[6]** Consider the traffic flow described by the following diagram. The arrows indicate the direction of one-way traffic.



(a) **[2]** Write down a linear system describing this traffic flow. **Do not perform any calculations at this stage.**

(b) **[2]** The reduced echelon form of the augmented matrix in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 50 \\ 0 & 1 & 1 & 0 & 1 & 150 \\ 0 & 0 & 0 & 1 & -1 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Give the general solution of the system (ignore the constraints on the variables for now).

(c) **[2]** If road BC is closed due to construction, what is the smallest flow possible along AC?

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