University of Ottawa

Department of Mathematics and Statistics

MAT 1302B: Mathematical Methods II

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Final Exam – April 23, 2009

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(1) You have 180 minutes to complete this exam.
(2) This exam consists of two parts. Questions 1 through 10 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 11 through 20 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(3) Each answer only question is worth two points. For the long answer questions, the number of points available for each question is indicated in square brackets.
(4) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(5) Write your student number at the top of each page in the space provided.
(6) No notes, books, calculators or scrap paper are allowed.
(7) The final page of the exam may be used for scrap work.
(8) Good luck!

Please do not write in the table below.

Question	1–10	11	12	13	14	15	16	17	18	19	20	Total
Maximum	20	4	4	4	4	4	5	8	6	5	6	70
Score												

Part A: Answer Only Questions

For Questions 1–10, only your final answer will be considered for marks. Each question is worth 2 points.

1. If

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$

what is AB?

Answer: $AB = \begin{bmatrix} 0 & -2 & 1 \\ 6 & 4 & -1 \\ 3 & 3 & -1 \end{bmatrix}$

2. If $z = 1 + 2i$ and $w = 3 - 2i$, compute $\bar{z}w$. Write your answer in the form $a + bi$ where a and b are real numbers.

Answer: $-1 - 8i$

3. Write the complex number $\frac{2-3i}{1+2i}$ in the form $a + bi$ where a and b are real numbers.

Answer: $\frac{-4}{5} - \frac{7}{5}i$

4. If $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 3$, what is $\det \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$?

Answer: -6

5. What are the eigenvalues of the matrix

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 10 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & -7 & 6 & 8 & 0 \\ 2 & -4 & 6 & 3 & 10 \end{bmatrix}?$$

Answer: $6, -1, 0, 8, 10$

6. Which of the following sets are subspaces of $\mathbb{R}^n$ for the given value of n?
   (a) $\{(0,0)\}$, $n = 3$
   (b) $\text{Span}\{(1,0,-1),(2,2,0)\}$, $n = 3$
   (c) $\{(x,x,y,-y) \mid x,y \in \mathbb{R}\}$, $n = 4$
   (d) $\{(w,x,y,z) \mid w,x,y,z \in \mathbb{R},\ 2x + 3y - z = 1\}$, $n = 4$
   (e) $\{(w,x,y,z) \mid w,x,y,z \in \mathbb{R},\ 2x + 3y + z = 0\}$, $n = 4$

Answer: (b), (c), (e)

7. Suppose P, Q and R are 3×3 matrices with

$$\det P = -1, \quad \det Q = 3, \quad \det R = 5.$$

What is $\det(2P^5 Q^T R Q^{-1} R^T)$?

Answer: -200

8. Which of the following statements are true for an $n \times n$ matrix M? Note that more than one statement may be correct (you should indicate **all** of the correct statements).

(a) $\det M = \det M^T$.
(b) M and M^T can have different eigenvalues.
(c) For every $\vec{b} \in \mathbb{R}^n$, the equation $M\vec{x} = \vec{b}$ has a solution.
(d) The equation $M\vec{x} = \vec{0}$ always has a solution.
(e) M is invertible if and only if zero is not an eigenvalue of M.
(f) M has at most n eigenvalues.

Answer: (a), (d), (e), (f)

9. Which of the following statements are true? Note that more than one statement may be correct (you should indicate **all** of the correct statements).

(a) It is possible for a set of 5 vectors in $\mathbb{R}^4$ to be linearly independent.
(b) Any basis of $\mathbb{R}^5$ must consist of 5 vectors.
(c) Any subspace contains the zero vector.
(d) Any basis of a subspace contains the zero vector.
(e) It is possible for a set of 4 vectors to span $\mathbb{R}^3$.
(f) A subspace can have more than one basis.

Answer: (b), (c), (e), (f)

10. What is the rank of

$$\begin{bmatrix} 0 & 8 & 0 & 9 & 10 & 5 & 6 & 3 \\ 0 & 0 & -1 & 0 & 8 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & -2 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}?$$

Answer: 4

Part B: Long Answer Questions

For Questions 11–20, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

11. **[4]** Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix}$.

Solution: $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} = (-7)\begin{vmatrix} 3 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \end{vmatrix} = (-7)(-1)\begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 5 & 0 \end{vmatrix} =$
$(-7)(-1)(-5)\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = -175.$

12. [**4**] Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \end{bmatrix}$ linearly dependent? If so, find a linear dependence relation.

Solution: We need to find out if the homogeneous system $A\vec{x} = \vec{0}$ has a nontrivial solution, where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 9 & -1 & 7 & 8 \end{bmatrix}$. So we row reduce (ignoring the augmented column of zeros):

$$A \xrightarrow[-9R_1+R_4\to R_4]{-R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -10 & 7 & 8 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -10 & 7 & 8 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-7R_3+R_2\to R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -10 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since x_4 is a free variable, the system has an infinite number of solutions and so the vectors are linearly dependent. To find a linear dependence relation, we must find a nontrivial solution: Take $x_4 = 1$. Since $x_3 + 2x_4 = 0$, we have $x_3 = -2$. Since $-10x_2 - 6x_4 = 0$ and $x_1 + x_2 = 0$, we have $x_2 = -6/10$ and $x_1 = 6/10$. Thus

$$\frac{6}{10}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 9 \end{bmatrix} - \frac{6}{10}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \end{bmatrix} = 0.$$

13. [**4**] Is the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ invertible? If so, find its inverse A^{-1}.

Solution: $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$
$\xrightarrow[-R_2\to R_2]{-R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$
$\xrightarrow[-R_3+R_1\to R_2]{-R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$. Thus we see that A is invertible and

$$A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

14. [**4**] Find the value(s) of t for which $\begin{bmatrix} 3 \\ 3 \\ 7 \\ t \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -5 \\ 0 \\ 2 \end{bmatrix}$.

Solution: We reduce the augmented matrix whose columns are the given vectors:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 3 & 5 & -5 & 3 \\ -4 & -1 & 0 & 7 \\ 1 & -2 & 2 & t \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & -2 & -6 \\ 0 & 3 & -4 & 19 \\ 0 & -3 & 3 & -3+t \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & -4 & 19 \\ 0 & -3 & 3 & -3+t \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & 28 \\ 0 & 0 & 0 & -12+t \end{array}\right].$$

This system has a solution (and so the vector $(3, 3, 7, t)$ is a linear combination of the other vectors) when the rightmost column is not a pivot column, that is, when $t = 12$.

15. [**4**] Find a basis for the space of solutions of the following system:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & + & 4x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 7x_3 & + & 10x_4 & = & 0 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -4 & 7 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The variables x_1 and x_3 are basic variables, and x_2 and x_4 are free variables. We have $x_1 = 2x_2 + 2x_4$ and $x_3 = -2x_4$.

Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$

Thus, a basis of the space of solutions is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

16. [**5**] Consider an economy with two sectors: work and leisure. In order to produce one unit, the work sector uses $\frac{1}{3}$ of a unit from the work sector and $\frac{1}{6}$ of unit from the leisure sector. To produce one unit, the leisure section uses $\frac{1}{6}$ of a unit of each of the work and leisure sectors.

(a) [**1**] Write the consumption matrix C corresponding to this economy.

Solution:

$$C = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

(b) [**1**] State the Leontief input-output equation relating the production vector $\vec{x}$ and the final demand vector $\vec{d}$.

Solution:

$$\vec{x} = C\vec{x} + \vec{d}$$

(c) [**3**] Find the production $\vec{x}$ needed to satisfy a final demand of $\vec{d} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$.

Solution: From the Leontief input-output equation, we see that

$$\vec{x} = (I - C)^{-1}\vec{d}.$$

We apply the formula for the inverse of a 2×2 matrix to

$$(I - C) = \begin{bmatrix} \frac{2}{3} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{5}{6} \end{bmatrix}$$

to obtain

$$\begin{aligned} (I - C)^{-1} &= \frac{1}{\det(I - C)} \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \\ &= \frac{6}{19} \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}. \end{aligned}$$

Thus

$$\vec{x} = \frac{6}{19} \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \end{bmatrix} = \frac{6}{19} \begin{bmatrix} 550 \\ 300 \end{bmatrix}.$$

17. [**8**] Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) [**3**] Use the characteristic equation to find the eigenvalues of A.

Solution: Expanding along the first column, we have

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} \\
&= -(1+\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1-\lambda \\ 1 & 1 \end{vmatrix} \\
&= -(1+\lambda)((1+\lambda)^2 - 1) - (-1-\lambda-1) + (1+1+\lambda) \\
&= -(1+\lambda)((1+\lambda)^2 - 1) + 2(\lambda+2) \\
&= -(1+\lambda)(\lambda^2 + 2\lambda) + 2(\lambda+2) \\
&= -(1+\lambda)(\lambda+2)\lambda + 2(\lambda+2) \\
&= -(\lambda+2)(\lambda^2 + \lambda - 2) \\
&= -(\lambda+2)^2(\lambda-1).
\end{aligned}$$

Thus the eigenvalues are -2 and 1.

(b) [**2**] Find a basis of the eigenspace of eigenvalue -2.

Solution: We row reduce the matrix $A + 2I$ to find the general solution to the homogenous system $(A + 2I)\vec{x} = \vec{0}$:

$$A + 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A basis of the eigenspace is thus $\{(-1, 1, 0), (-1, 0, 1)\}$.

(c) [**2**] Find a basis of the eigenspace of eigenvalue 1.

Solution: We row reduce the matrix $A - I$ to find the general solution to the homogenous system $(A - I)\vec{x} = \vec{0}$:

$$A - I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A basis of the eigenspace is thus $\{(1, 1, 1)\}$.

(d) [**1**] Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

18. [**6**] Two telephone companies *OutOfOrder* and *NoService* are competing. A statistical study has shown that in each **6 month** period, 60% of the clients of OutOfOrder stay with the company while 40% of them which to NoService. During the same period, 20% of the clients of NoService switch to OutOfOrder, while 80% stay with NoService

On January 1, 2009, OutOfOrder had 50 thousand clients and NoService had 25 thousand clients.

(a) [**1**] Write the migration matrix M.

Solution:

$$M = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}.$$

(b) [**1**] How many clients will each company have on July 1, 2009?

Solution:

$$\begin{aligned} \vec{x}_1 &= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 50\,000 \\ 25\,000 \end{bmatrix} \\ &= \begin{bmatrix} 35\,000 \\ 40\,000 \end{bmatrix}. \end{aligned}$$

Thus, OutOfOrder will have 35 000 customers and NoService will have 40 000 customers.

(c) [**1**] How many clients will each company have on January 1, 2010?

Solution:

$$\begin{aligned} \vec{x}_2 &= \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 35\,000 \\ 40\,000 \end{bmatrix} \\ &= \begin{bmatrix} 29\,000 \\ 46\,000 \end{bmatrix}. \end{aligned}$$

Thus, OutOfOrder will have 29 000 customers and NoService will have 46 000 customers.

(d) [**3**] Assuming the migration matrix M stays constant in the long term, find the market share (i.e. percentage of the total customers) of each company in the long term. That is, find the equilibrium market shares.

Solution: Since M is a regular stochastic matrix, the equilibrium is given by an eigenvector with eigenvalue one. So we must solve

$$M\vec{q} = \vec{q} \qquad \text{or} \qquad (M - I)\vec{q} = \vec{q}.$$

We row reduce:

$$M - I = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}.$$

Thus, a basis for the eigenspace is $\{(1, 2)\}$. Thus, there exists a scalar $\alpha \neq 0$ such that

$$(q_1, q_2) = \alpha(1, 2).$$

Since the market shares must add up to one, we have

$$1 = q_1 + q_2 = \alpha(1 + 2) = 3\alpha \implies \alpha = \frac{1}{3}.$$

Therefore

$$\vec{q} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

Thus, in the long term, OutOfOrder has 1/3 of the market and NoService has 2/3 of the market.

19. [**5**] Consider the matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}$, whose reduced echelon form is $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) [**2**] Find a basis of Nul A.

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

and so a basis is $\{(2,1,0,0),(1,0,-2,1)\}$.

(b) [**1**] Find the dimension of Nul A.

Solution: 2.

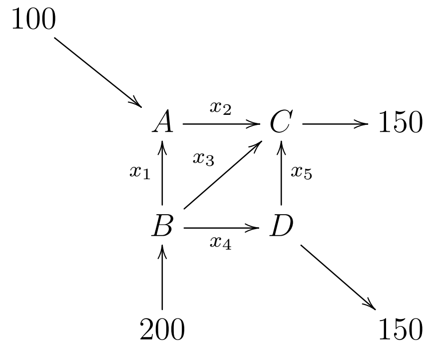
(c) [**1**] Find a basis of Col A.

Solution: $\{(-3,1,2),(-1,2,5)\}$.

(d) [**1**] Find the dimension of Col A.

Solution: 2.

20. [**6**] Consider the traffic flow described by the following diagram. The arrows indicate the direction of one-way traffic.



(a) [**2**] Write down a linear system describing this traffic flow. **Do not perform any calculations at this stage.**

Solution: Setting the total flow in equal to the total flow out at each intersection, we obtain the following linear system:

Intersection	**Incoming**		**Outgoing**
A	$100 + x_1$	$=$	x_2
B	200	$=$	$x_1 + x_3 + x_4$
C	$x_2 + x_3 + x_5$	$=$	150
D	x_4	$=$	$x_5 + 150$

(b) **[2]** The reduced echelon form of the augmented matrix in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 50 \\ 0 & 1 & 1 & 0 & 1 & 150 \\ 0 & 0 & 0 & 1 & -1 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Give the general solution of the system (ignore the constraints on the variables for now).

Solution: The basic variables are x_1, x_2 and x_4; and the free variables are x_3 et x_5. Writing the basic variables in terms of the free variables, we obtain the general solution to the system:

$$\begin{aligned} x_1 &= -x_3 - x_5 + 50 \\ x_2 &= -x_3 - x_5 + 150 \\ x_3 &\text{ free} \\ x_4 &= x_5 + 150 \\ x_5 &\text{ free} \end{aligned}$$

(c) **[2]** If road BC is closed due to construction, what is the smallest flow possible along AC?

Solution: We take $x_3 = 0$. Furthermore, the constraints $x_i \geq 0$ imply

$$\begin{aligned} x_1 \geq 0 &\implies x_5 \leq 50 \\ x_2 \geq 0 &\implies x_5 \leq 150 \\ x_4 \geq 0 &\implies x_5 \geq -150 \end{aligned}$$

and so $0 \leq x_5 \leq 50$. We conclude that the minimum flow possible along AC is

$$x_2 = -50 + 150 = 100.$$

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