University of Ottawa

Department of Mathematics and Statistics

Math 1302A: Mathematical Methods II
Instructor: Alistair Savage

Final Exam – April 15, 2008
Solutions

Surname ____________________ First Name ____________________

Student # ____________________

Instructions:

(1) You have 180 minutes to complete this exam.
(2) This exam consists of two parts. Questions 1 through 13 are **answer only**. For these questions, only your final answer will be considered for marks. Questions 14 through 22 are **long answer**. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(3) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(4) Write your student number at the top of each page in the space provided.
(5) No notes, book, calculators or scrap paper are allowed.
(6) The final page of the exam (page 15) may be used for scrap work.
(7) Good luck!

Please do not write in the table below.

Question	1–13	14	15	16	17	18	19	20	21	22	Total
Maximum	18	3	6	5	4	5	6	3	3	7	60
Score											

Part A: Answer Only Questions

For Questions 1–13, only your final answer will be considered for marks.

1. **(1 point)** Which of the following statements is true for every matrix M with m rows and n columns, where $m < n$?

   (a) The system $M\mathbf{x} = \mathbf{0}$ has no solution.
   (b) The system $M^T\mathbf{x} = \mathbf{0}$ has no solution.
   (c) The system $M\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
   (d) For every $\mathbf{b}$ in $\mathbb{R}^m$, the system $M\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
   (e) For some $\mathbf{b}$ in $\mathbb{R}^m$, the system $M\mathbf{x} = \mathbf{b}$ has a unique solution.
   (f) For every $\mathbf{b}$ in $\mathbb{R}^n$, the system $M^T\mathbf{x} = \mathbf{b}$ has at least one solution.

   Answer: (c)

2. **(1 point)** Which of the following identities is **not** always true for $n \times n$ matrices M and N?

   (a) $M + N = N + M$,
   (b) $c(M + N) = cM + cN$,
   (c) $(c + d)M = cM + dN$,
   (d) $MN = NM$,
   (e) $c(dM) = (cd)M$,
   (f) $c(MN) = (cM)N$.

   Answer: (d)

3. **(2 points)** Which of the following sets are subspaces of $\mathbb{R}^3$? More than one answer may be correct.

   (a) $\{(0, 0, 0)\}$
   (b) $\{(a, b, 2) \mid a, b \in \mathbb{R}\}$
   (c) $\{(a^2, 0, 0) \mid a \in \mathbb{R}\}$
   (d) $\{(a^3, 0, b) \mid a, b \in \mathbb{R}\}$
   (e) $\{(a, b, c) \mid a, b, c \in \mathbb{R},\ a + 2b + 3c = 0\}$
   (f) $\{(a, b, c) \mid a, b, c \in \mathbb{R},\ a + 2b + 3c = 1\}$

   Answer: (a), (d) and (e)

4. **(1 point)** Of the following matrices, circle those which are in reduced row echelon form.

$$\begin{bmatrix}1&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \quad \begin{bmatrix}1&0&0&1\\0&1&0&0\\0&0&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&0\\0&1&0&1\\0&0&0&0\end{bmatrix}, \quad \begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&1&0\end{bmatrix}, \quad \begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix},$$
$$\begin{bmatrix}1&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&0&1\\0&0&1&0\\0&0&0&0\end{bmatrix}, \quad \begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{bmatrix}.$$

Answer: First two in first row, first and third in second row, second in third row.

5. **(3 points)** If

$$M = \begin{bmatrix}1&3&2\\2&-1&0\\-3&2&-2\end{bmatrix}, \quad \text{and} \quad N = \begin{bmatrix}1&-2\\2&1\\1&3\end{bmatrix},$$

compute MN.

Answer: $MN = \begin{bmatrix}9&7\\0&-5\\-1&2\end{bmatrix}$

6. **(1 point)** Compute $(2-3i)(-1+i)$. Write your answer in the form $a+bi$ where a and b are real numbers.

Answer: $1+5i$

7. **(2 points)** Find real numbers a and b such that

$$a + bi = \frac{1+2i}{1-i}.$$

Answer: $a = -1/2$, $b = 3/2$

8. **(1 point)** Suppose $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = -2$. What is $\begin{vmatrix} 2b & 2d \\ -2a & -2c \end{vmatrix}$?

Answer: -8

9. **(1 point)** Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 3 & -2 & -1 & -2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

What is $\det(MN)$?

Answer: -8

10. **(1 point)** Suppose

$$M = \begin{bmatrix} 1 & -6 & 8 & 10 & -5 & 12 & 0 \\ 0 & 0 & 4 & 5 & 7 & -3 & -2 \\ 0 & 0 & 0 & -2 & 0 & 8 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the rank of M?

Answer: 3

11. **(1 point)** For the matrix M of the previous question, what is the dimension of the null space of M?

Answer: 4

12. **(2 points)** Suppose A, B and C are 6×6 matrices with

$$\det A = 2, \quad \det B = 7, \quad \det C = -1.$$

What is $\det(-AB^3CC^TA^TB^{-3})$?

Answer: 4

13. **(1 point)** What are the eigenvalues of the matrix

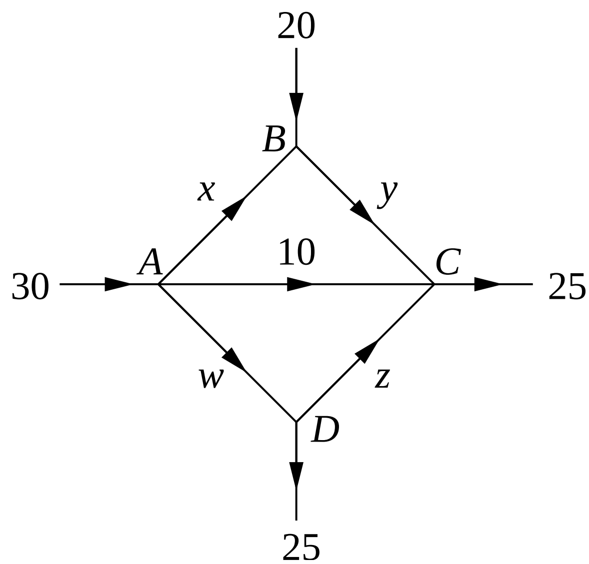
$$\begin{bmatrix} 5 & 8 & 20 & -2 \\ 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 2 \end{bmatrix}?$$

Answer: 5, -1, 0, 2

Part B: Long Answer Questions

For Questions 14–22, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

14. **(3 points)** Write the linear system describing the following traffic problem, labeling each equation by the intersection (A, B, C, or D) to which it corresponds. **Do not solve the system.**



Solution:

$$\begin{aligned} A: &\quad 30 = x + w + 10 \\ B: &\quad 20 + x = y \\ C: &\quad 10 + y + z = 25 \\ D: &\quad w = 25 + z \end{aligned}$$

15. **(6 points)** Suppose the augmented matrix of a linear system is

$$\left[\begin{array}{ccc|c} 1 & k-1 & 1 & 0 \\ 0 & 1 & k+1 & 1 \\ 0 & 0 & k^2-1 & k-1 \end{array}\right].$$

Determine the value(s) of k for which this system has

(a) no solution,
(b) infinitely many solutions,
(c) a unique solution.

Work / Justification:

If $k \neq \pm 1$, then the system has three pivots and thus has a unique solution. If $k = 1$, the bottom row is all zeros. Thus the system has one free variable and so has infinitely many solutions. If $k = -1$, the bottom row has a pivot in the last column. Thus the system has no solution.

Final Answer: a) -1 b) 1 c) $k \neq \pm 1$

16. **(5 points)** Are the vectors $(1, 1, 2)$, $(1, 2, -3)$, and $(-1, 0, -7)$ linearly independent? If not, find a non-trivial linear combination of these three vectors that is equal to $\mathbf{0}$.

Solution:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 2 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix has only two pivots and thus the vectors are linearly dependent. Any solution to the corresponding homogeneous equation gives a linear combination that is equal to zero. Thus we have

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 & \quad \text{free} \end{aligned}$$

We can take $x_3 = 1$, which gives $x_1 = 2$ and $x_2 = -1$. Then we see that

$$2(1, 1, 2) - (1, 2, -3) + (-1, 0, -7) = (0, 0, 0).$$

17. **(4 points)** Find the inverse of the matrix

$$M = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}.$$

Solution:

$$\left[\, M \mid I \,\right] = \left[\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right]$$

Thus

$$M^{-1} = \begin{bmatrix} -3/2 & -3/2 & 11/2 \\ 1/2 & 1/2 & -3/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}.$$

18. **(5 points)** Suppose an economy with two sectors has unit consumption matrix

$$C = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}$$

and final demand vector

$$\mathbf{d} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}.$$

Determine the production levels necessary to satisfy the given final demand. (Hint: use the inverse matrix method instead of row reduction.)

Solution: We must solve

$$(I - C)\mathbf{x} = \mathbf{d} \Longleftrightarrow \mathbf{x} = (I - C)^{-1}\mathbf{d}.$$

Now

$$I - C = \begin{bmatrix} 0.9 & -0.4 \\ -0.3 & 0.8 \end{bmatrix} \Rightarrow (I - C)^{-1} = \frac{1}{0.6} \begin{bmatrix} 0.8 & 0.4 \\ 0.3 & 0.9 \end{bmatrix}.$$

Therefore

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} = \frac{1}{0.6} \begin{bmatrix} 0.8 & 0.4 \\ 0.3 & 0.9 \end{bmatrix} \begin{bmatrix} 15 \\ 12 \end{bmatrix} = \frac{1}{0.6} \begin{bmatrix} 16.8 \\ 15.3 \end{bmatrix} = \begin{bmatrix} 28 \\ 25.5 \end{bmatrix}$$

19. **(6 points)** Find the eigenvalues of the matrix

$$M = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}.$$

For each eigenvalue, find the corresponding eigenspace.

Solution: To find the eigenvalues, we solve the characteristic equation.

$$\begin{aligned} 0 &= \det(M - \lambda I) \\ &= \begin{vmatrix} 7-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} = (7-\lambda)(2-\lambda) + 4 = \lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6). \end{aligned}$$

Thus the eigenvalues are 3 and 6.

To find the eigenspace corresponding to $\lambda = 3$, we row reduce

$$\left[\ (M - 3I) \mid \mathbf{0}\ \right] = \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 4 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus the eigenspace is $\text{Span}\{(1,4)\}$.

To find the eigenspace corresponding to $\lambda = 6$, we row reduce

$$\left[\ (M - 6I) \mid \mathbf{0}\ \right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus the eigenspace is $\text{Span}\{(1,1)\}$.

20. **(3 points)** The eigenvalues of the matrix

$$M = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

are 1 and 2. The eigenspace corresponding to the eigenvalue 1 is

$$\left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix},\ t \in \mathbb{R} \right\},$$

and the eigenspace corresponding to the eigenvalue 2 is

$$\left\{ \begin{bmatrix} r \\ s \\ -3r + 3s \end{bmatrix},\ r, s \in \mathbb{R} \right\}.$$

Find a matrix P such that the matrix $P^{-1}MP$ is diagonal.

Solution: A basis for the eigenspace with eigenvalue 1 is $\{(1, 1, 1)\}$ and a basis for the eigenspace with eigenvalue 2 is $\{(1, 0, -3), (0, 1, 3)\}$. Therefore, the desired matrix is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

21. **(3 points)** Let P and M be 2×2 matrices with

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

If

$$P^{-1}MP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

find M^7. Note: You do not need to compute powers of scalars (that is, you can leave expressions like 2^n in your answer).

Answer: We have

$$P^{-1} = \frac{1}{2(-2) - 1(1)} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Thus

$$\begin{aligned} M^7 &= P \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^7 P^{-1} \\ &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2^7 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2^8 & 2^7 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2^9 - 1 & 2^8 + 2 \\ 2^8 + 2 & 2^7 - 4 \end{bmatrix} \end{aligned}$$

22. In a certain country, the mobile phone industry is dominated by two companies: Ten-Fours and Siren. Ten-Fours has 180 000 customers and Siren has 120 000 customers. Assume that, every year, 10% of the customer base of Ten-Fours switches to Siren and 5% of the customer base of Siren switches to Ten-Fours. For the purposes of this question, suppose no customer leaves a company without switching to the other one and no company attracts customers that are not leaving the other (that is, the only changes in customer base come from switching between the two companies).

(a). **(1 point)** Write down the transition (migration) matrix M and initial state vector $\mathbf{x}_0$ for this problem.

Solution:

$$M = \begin{bmatrix} .9 & .05 \\ .1 & .95 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 180\,000 \\ 120\,000 \end{bmatrix}.$$

(b). **(2 points)** Find the number of customers of Ten-Fours after one year.

Solution:

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .9 & .05 \\ .1 & .95 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .56 \\ .44 \end{bmatrix}$$

Therefore, after one year, Ten-Fours will have $(0.56)(300\,000) = 168\,000$ customers.

(c). **(4 points)** Find the number of customers of Ten-Fours after many years. That is, find the long term stable number of customers of Ten-Fours.

Solution: Since M is a regular stochastic matrix (all of its entries are strictly positive), the state will approach the unique steady-state vector $\mathbf{q}$. To find $\mathbf{q}$, we find the eigenspace with eigenvalue 1.

$$\left[\ (M-I) \mid \mathbf{0}\ \right] = \left[\begin{array}{cc|c} -.1 & .05 & 0 \\ .1 & -.05 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Since we want $\mathbf{q}$ to be a probability vector, we choose x_2 so that the entries sum to one. So

$$x_3 = \left(\frac{1}{2} + 1\right)^{-1} = \frac{2}{3}.$$

Therefore

$$\mathbf{q} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

Thus, in the long term, Ten-Fours has $\frac{1}{3}(300\,000) = 100\,000$ customers.

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