University of Ottawa

Department of Mathematics and Statistics

MAT 1302D: Mathematical Methods II

Instructor: Erhard Neher

Final Exam ; April 28, 2006

Family Name: ______________________________

First Name: ______________________________

Student number: ___________________________

Some Advice

Take 5 minutes to read the entire paper before you begin to write, and read each question carefully. Remember that the multiple choice questions are only worth 1 point and the others are worth between 6 and 8 points. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given. Do read the instructions below.

Please read these instructions carefully:

- You have 180 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. **The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.**
- Questions 1 to 10 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the space provided.
- Questions 11-15 require a complete solution. The number of points each part is worth is indicated in square brackets, so spend your time accordingly. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.**
- Where it is possible to check your work, do so.

Good luck!

Quest.	1 – 10	11.	12.	13.	14.	15.	Total
maximal	10	7	6	7	7	8	45
score							

1. For a non-homogeneous system of 150 equations in 100 unknowns, answer the following three questions:
- ∘ Can the system be inconsistent?
- ∘ Can the system have infinitely many solutions?
- ∘ Can the system have exactly one solution?

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

My answer:________________

2. Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row of A^{-1} is $(-1, -1, 1)$.
C. The second row of A^{-1} is $(1, 2, -1)$.
D. The first row of A^{-1} is $(2, 0, -1)$.
E. The second column of A^{-1} is $(0, 2, -1)^T$.
F. All of B, C, D, E are true.

My answer:________________

3. If A is an $n \times n$ matrix, which of the following statements are (always) true?

I. If $Ax = 0$ has a unique solution, then $\det A \neq 0$.

II. If the columns of A span $\mathbb{R}^n$, the columns of A are linearly **independent**.

III. If $Ax = b$ is not consistent for every $b \in \mathbb{R}^n$, then $\operatorname{rank} A = n$.

IV. If $Ax = b$ is consistent for every $b \in \mathbb{R}^n$, then A is invertible.

A. I, II and III

B. I, II, III and IV

C. I, and IV

D. I, II, and IV

E. I, and III

F. II, III and IV

My answer:________________

4. Find the value of t for which $\begin{bmatrix} 3 \\ 3 \\ 7 \\ t \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -5 \\ 0 \\ 2 \end{bmatrix}$.

A. -1
B. 3
C. 6
D. 10
E. 12
F. 14

My answer:________________

5. If A is an $n \times 2$ matrix and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then the second column of the matrix AB is

A. not defined unless $n = 2$.

B. the same as the second column of A.

C. the same as the second column of B.

D. the same as the first column of A.

E. the same as the first column of B.

F. the sum of the first and the second column of A.

My answer:________________

6. Which two of the following are subspaces of $\mathbb{R}^3$?

$S = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$

$T = \{(2x, 2x - y, y) \mid x, y \in \mathbb{R}\}$

$U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 1\}$

A. S and T.
B. T and V.
C. T and U.
D. S and V.
E. S and U.
F. U and V.

My answer:________________

7. Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix}$.

A. -135

B. -105

C. 165

D. -205

E. -175

F. 225

My answer:________________

8. Compute $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}^{2006}$. (Hint: Use the partition $\begin{bmatrix} 1 & 0 & | & 0 & 0 \\ 0 & 1 & | & 0 & 0 \\ - & - & - & - & - \\ 0 & 0 & | & 1 & 0 \\ -2 & 0 & | & 0 & 1 \end{bmatrix}$)

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4012 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4012 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2^{2006} & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2006^{-2} & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4012 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4012 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

My answer:________________

9. The dimension of the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\0\\9\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\\7\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\\8\end{bmatrix}$ is:

A. 5
B. 4
C. 3
D. 2
E. 1
F. 0

My answer:________________

10. A basis for the solution space of the system

$$\begin{array}{rcrcrcrcl} u & - & 2x & + & 3y & + & 4z & = & 0 \\ 2u & - & 4x & + & 7y & + & 10z & = & 0 \end{array}$$

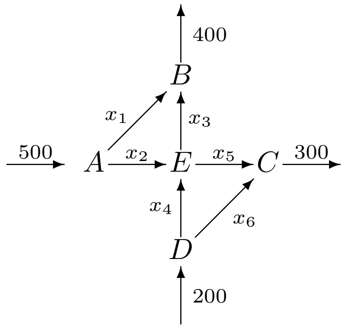
is:

A. $\{(0,\ 0,\ 0,\ 0)\}$
B. $\{(2,\ 1,\ 0,\ 0)\}$
C. $\{(1,\ 2,\ 0,\ 0)\}$
D. $\{(2,\ 1,\ 0,\ 0),\ (1,\ -3,\ -4,\ 1)\}$
E. $\{(2,\ 1,\ 0,\ 0),\ (2,\ 0,\ -2,\ 1)\}$
F. $\{(2,\ 0,\ -2,\ 1)\}$

For reasons of space in the answers above a vector $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ has been written in the form $(a,\ b,\ c,\ d)$.

My answer:________________

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i ($i = 1, \ldots 6$ denotes the unknown number of cars which passed along the indicated streets during the same period.



[1] a) Write down the linear system in the variables $x_1, x_2, \ldots x_6$ which describes the traffic flow. (*Do not perform any operations on your equations: this is done for you in* (b))

[3] b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ 0 & 1 & -1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & 1 & 200 \\ 0 & 0 & 0 & 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution.

(continued on next page)

[2] c) If the flow along DE is limited to at most 100 cars per minute because of roadwork, what is the maximum possible flow along EC?

[1] d) If you are given at most two extra observers, along which streets would you place them so as to determine the exact traffic flows along each street?

Bonus

[1] e) Give an example of two streets where observers could be placed, but one would still not be able to determine the exact flows along each street.

12. An open economy has two sectors, Work and Leisure. To produce one unit of output, the Work sector requires $\frac{1}{2}$ of a unit of Work and $\frac{1}{3}$ of a unit of Leisure, while the Leisure sector requires $\frac{1}{4}$ of a unit of Work and $\frac{1}{3}$ of a unit of Leisure to produce one unit of its own output.

[2] a) Give the input-output or consumption matrix C for this economy.

[1] b) Give the Leontief Input-Output Model equation, relating the production vector x, the final demand vector d and the input-output or consumption matrix C.

[3] c) What is the production vector necessary to satisfy the final demand vector $d = \begin{bmatrix} 600 \\ 210 \end{bmatrix}$?

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13. Two telephone service providers LocoTel and FouPhone compete for customers. Market research has shown that in any 3-month period, 80% of LocoTel's customers stay with them, while 20% switch to FouPhone. In the same period, 30% of FouPhone's customers switch to LocoTel, while 70% remain with FouPhone.

On January 1, 2006, LocoTel has 40 thousand customers and FouPhone has 60 thousand, i.e. the initial customer vector is $X_0 = 10^3 \begin{bmatrix} 40 \\ 60 \end{bmatrix}$. Assume that the total number of customers does not increase during a 6 month period.

[1] a) Give the customer 'migration' matrix M.

[2] b) What is the expected number of customers each company has on April 1, 2006?

[2] c) What is the expected number of customers each company has on July 1, 2006?

(continued on next page)

[2] d) Assuming the customer migration matrix M remains the same for a long period, find the percentages of customers using LocoTel and FouPhone in the long-term. That is, solve $MX = X$ to determine the long-term or stable customer distribution X, remembering that the entries of X must add to $1 = 100\%$.

14. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & -3 & -2 \\ 1 & -2 & 0 \end{bmatrix}$.

[2] a) Compute the characteristic polynomial to show that the eigenvalues of A are -5 and 1.

[1] b) Find a basis of $E_{-5} = \{v \in \mathbb{R}^3 \mid Av = -5v\}$.

[1] c) Find a basis of $E_1 = \{v \in \mathbb{R}^3 \mid Av = v\}$.

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[2] d) Find an invertible matrix P, and a diagonal matrix D such that $P^{-1}AP = D$.

[1] e) Check that your choice of P in part (d) is invertible.

[1] f) **(bonus)** Find $A^{100}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

15. Consider the matrix $A = \begin{bmatrix} 2 & 4 & -2 & 1 & -1 \\ -2 & -5 & 7 & 3 & 10 \\ 3 & 7 & -8 & 6 & -2 \end{bmatrix}$.

[2] a) Show that the reduced row echelon form of A is

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 9 & 0 & 9 \\ 0 & 1 & -5 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

[3] b) Give a basis for the null space $\operatorname{Nul} A$ of the matrix A.

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[1] c) What is the dimension of $\operatorname{Nul} A$?

[1] d) Give a basis for the column space of A.

[1] e) Do the columns of A span $\mathbb{R}^3$?