1. For a non-homogeneous system of 150 equations in 100 unknowns, answer the following three questions:
   - Can the system be inconsistent? Yes
   - Can the system have infinitely many solutions? Yes
   - Can the system have exactly one solution? Yes

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

My answer: B

2. Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row of A^{-1} is $(-1,-1,1)$.
C. The second row of A^{-1} is $(1,2,-1)$.
D. The first row of A^{-1} is $(2,0,-1)$.
E. The second column of A^{-1} is $(0,2,-1)^T$.
F. All of B, C, D, E are true.

My answer: B

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}, \text{ hence}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

3. If A is an $n \times n$ matrix, which of the following statements are (always) true?

I. If $Ax = 0$ has a unique solution, then $\det A \neq 0$. True

II. If the columns of A span $\mathbb{R}^n$, the columns of A are linearly independent. True

III. If $Ax = b$ is not consistent for every $b \in \mathbb{R}^n$, then rank $A = n$. False

IV. If $Ax = b$ is consistent for every $b \in \mathbb{R}^n$, then A is invertible. True

A. I, II and III

B. I, II, III and IV

C. I, and IV

D. I, II, and IV

E. I, and III

F. II, III and IV

My answer: D

4. Find the value of t for which $\begin{bmatrix} 3 \\ 3 \\ 7 \\ t \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -5 \\ 0 \\ 2 \end{bmatrix}$.

A. -1
B. 3
C. 6
D. 10
E. 12
F. 14

My answer: E

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 3 & 5 & -5 & 3 \\ -4 & -1 & 0 & 7 \\ 1 & -2 & 2 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 2 & -2 & -6 \\ 0 & 3 & -4 & 19 \\ 0 & -3 & 3 & t-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & -4 & 19 \\ 0 & -3 & 3 & t-3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 28 \\ 0 & 0 & 0 & t-12 \end{bmatrix}$$

. Thus, a solution exists $\Leftrightarrow$ the system is soluble $\Leftrightarrow$ $t = 12$

5. If A is an $n \times 2$ matrix and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then the second column of the matrix AB is

A. not defined unless $n = 2$.

B. the same as the second column of A.

C. the same as the second column of B.

D. the same as the first column of A.

E. the same as the first column of B.

F. the sum of the first and the second column of A.

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}+a_{12} & a_{11} \\ a_{21}+a_{22} & a_{21} \\ \vdots & \vdots \end{bmatrix}$$

My answer: D

6. Which two of the following are subspaces of $\mathbb{R}^3$?

$S = \{(x,y,z) \in \mathbb{R}^3 \mid xy = 0\}$ No: $(1,0,0) \in S$, $(0,1,0) \in S$ but $(1,0,0)+(0,1,0) = (1,1,0) \notin S$

$T = \{(2x, 2x-y, y) \mid x,y \in \mathbb{R}\}$ Yes, since $T = \{x(2,2,0) + y(0,-1,1) : x,y \in \mathbb{R}\}$ $= \text{Span}\{(2,2,0), (0,-1,-1)\}$

$U = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$ → Yes, since $U = \text{Null } A$ for $A = [1\ 1\ 1]$

$V = \{(x,y,z) \in \mathbb{R}^3 \mid x-y+z = 1\}$ → No, since $(0,0,0) \notin V$.

A. S and T.

B. T and V.

C. T and U.

D. S and V.

E. S and U.

F. U and V.

My answer: C

7. Compute the determinant $\begin{vmatrix} 0 & 0 & 0 & 7 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix}$.

A. -135
B. -105
C. 165
D. -205
E. -175
F. 225

$$\underset{\substack{\uparrow \\ \text{row 4}}}{=} -\begin{vmatrix} 0 & 0 & 0 & 7 \\ 3 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \end{vmatrix} \underset{\substack{\uparrow \\ \text{column 4}}}{=} -(-7)\begin{vmatrix} 3 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 5 & 0 \end{vmatrix} \underset{\substack{\uparrow \\ \text{last row}}}{=} -5\cdot 7\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= -35(3+2) = -35\cdot 5 = -175$$

My answer: E

8. Compute $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}^{2006}$. (Hint: Use the partition $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{array}\right]$)

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4012 & 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4012 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2^{2006} & 0 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2006^{-2} & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4012 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ F. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4012 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

My answer: B

Using matrix partition in 2x2 blocks:

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}\begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ A+B & I \end{bmatrix}$$, hence for the given matrix

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}^{2006} = \begin{bmatrix} I & 0 \\ 2006A & I \end{bmatrix}$$, with $2006A = \begin{bmatrix} 0 & 0 \\ -4012 & 0 \end{bmatrix}$

9. The dimension of the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}1\\1\\0\\9\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\\7\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\\8\end{bmatrix}$ is:

A. 5
B. 4
C. 3
D. 2
E. 1
F. 0

We determine the dim of the column space of

$$A = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 1 & 0 & 0\\0 & 0 & 1 & 2\\9 & -1 & 7 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 2\\0 & -10 & 7 & 8\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 1 & 0 & 0\\0 & -10 & 7 & 8\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

This is a ref of A, hence dim = rank of $A = 3$

My answer: C

10. A basis for the solution space of the system

$$\begin{array}{rcrcrcrcl} u & - & 2x & + & 3y & + & 4z & = & 0\\ 2u & - & 4x & + & 7y & + & 10z & = & 0 \end{array}$$

is:

A. $\{(0,\ 0,\ 0,\ 0)\}$
B. $\{(2,\ 1,\ 0,\ 0)\}$
C. $\{(1,\ 2,\ 0,\ 0)\}$
D. $\{(2,\ 1,\ 0,\ 0),\ (1,\ -3,\ -4,\ 1)\}$
E. $\{(2,\ 1,\ 0,\ 0),\ (2,\ 0,\ -2,\ 1)\}$
F. $\{(2,\ 0,\ -2,\ 1)\}$

For reasons of space in the answers above a vector $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ has been written in the form $(a,\ b,\ c,\ d)$.

$$\begin{bmatrix}1 & -2 & 3 & 4 & 0\\2 & -4 & 7 & 10 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3 & 4\\0 & 0 & 1 & 2\end{bmatrix}$$

My answer: E

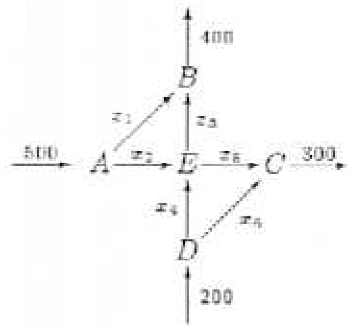
$$\sim \begin{bmatrix}1 & -2 & 0 & -2\\0 & 0 & 1 & 2\end{bmatrix}$$

Equivalent linear system: $u - 2x - 2z = 0$, $y + 2z = 0$

general solution $u = 2x + 2z$, $y = -2z$,

$$\begin{bmatrix}u\\x\\y\\z\end{bmatrix} = \begin{bmatrix}2x+2z\\x\\-2z\\z\end{bmatrix} = x\begin{bmatrix}2\\1\\0\\0\end{bmatrix} + z\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i $(i = 1, \ldots 6)$denotes the unknown number of cars which passed along the indicated streets during the same period.



[1] a) Write down the linear system in the variables $x_1, x_2, \ldots x_6$ which describes the traffic flow. (*Do not perform any operations on your equations: this is done for you in* (b))

(A) $x_1 + x_2 = 500$

(B) $x_1 + x_3 = 400$

(C) $x_5 + x_6 = 300$

(D) $x_4 + x_6 = 200$

(E) $x_2 - x_3 + x_4 - x_5 = 0$

[3] b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ 0 & 1 & -1 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & 1 & 200 \\ 0 & 0 & 0 & 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution.

$x_1 = 400 - x_3$

$x_2 = 100 + x_3$

$x_4 = 200 - x_6$

$x_5 = 300 - x_6$

x_5, x_6 are free variables

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[2] c) If the flow along DE is limited to at most 100 cars per minute because of roadwork, what is the maximum possible flow along EC?

We are given: $200 - x_6 = x_4 \leq 100$,

therefore $200 - 100 = 100 \leq x_6$ and then

$$x_5 = 300 - x_6 \leq 300 - 100 = 200,$$

i.e. the maximum flow is 200

[1] d) If you are given at most two extra observers, along which streets would you place them so as to determine the exact traffic flows along each street?

Since x_1, x_2, x_4 and x_5 are determined by x_3 and x_6, we need to place the extra observers so that we can determine x_3 and x_6, ie on the streets BE and CD

Bonus

[1] e) Give an example of two streets where observers could be placed, but one would still not be able to determine the exact flows along each street.

An observer on AB determines x_1, hence x_3.

A second observer on AE would not help to determine x_6.

12. An open economy has two sectors, Work and Leisure. To produce one unit of output, the Work sector requires $\frac{1}{2}$ of a unit of Work and $\frac{1}{3}$ of a unit of Leisure, while the Leisure sector requires $\frac{1}{4}$ of a unit of Work and $\frac{1}{3}$ of a unit of Leisure to produce one unit of its own output.

[2] a) Give the input-output or consumption matrix C for this economy.

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

[1] b) Give the Leontief Input-Output Model equation, relating the production vector x, the final demand vector d and the input-output or consumption matrix C.

$CX + d = X$, equivalently: $(I-C)X = d$

[3] c) What is the production vector necessary to satisfy the final demand vector $d = \begin{bmatrix} 600 \\ 210 \end{bmatrix}$?

$$[I-C \mid d] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 600 \\ -\frac{1}{3} & \frac{2}{3} & 210 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 2400 \\ -1 & 2 & 630 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -630 \\ 2 & -1 & 2400 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -630 \\ 0 & 3 & 3660 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -630 \\ 0 & 1 & 1220 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1810 \\ 0 & 1 & 1220 \end{bmatrix}$$

hence $X = \begin{bmatrix} 1810 \\ 1220 \end{bmatrix}$.

(Alternative solution: $(I-C)X = d \Leftrightarrow X = (I-C)^{-1}d$.

$$(I-C)^{-1} = \begin{bmatrix} 1/2 & -1/4 \\ -1/3 & 2/3 \end{bmatrix}^{-1} = \left(\tfrac{1}{2}\cdot\tfrac{2}{3} - \tfrac{1}{3}\cdot\tfrac{1}{4}\right)^{-1} \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix} =$$

$$= \left(\frac{4-1}{12}\right)^{-1} \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix} = 4 \begin{bmatrix} 2/3 & 1/4 \\ 1/3 & 1/2 \end{bmatrix} = \begin{bmatrix} 8/3 & 1 \\ 4/3 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 8/3 & 1 \\ 4/3 & 2 \end{bmatrix} \begin{bmatrix} 600 \\ 210 \end{bmatrix} = \begin{bmatrix} 1600 + 210 \\ 800 + 420 \end{bmatrix} = \begin{bmatrix} 1810 \\ 1220 \end{bmatrix}$$

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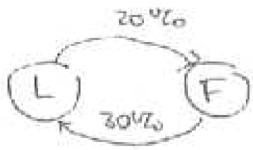
13. Two telephone service providers LocoTel and FouPhone compete for customers. Market research has shown that in any <u>3-month period</u>, 80% of LocoTel's customers stay with them, while 20% switch to FouPhone. In the same period, 30% of FouPhone's customers switch to LocoTel, while 70% remain with FouPhone.

On January 1, 2006, LocoTel has 40 thousand customers and FouPhone has 60 thousand, i.e. the initial customer vector is $X_0 = 10^3 \begin{bmatrix} 60 \\ 40 \end{bmatrix}$. Assume that the total number of customers does not increase during a 6 month period.

[1] a) Give the customer 'migration' matrix M.



$$M = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

[2] b) What is the expected number of customers each company has on April 1, 2006?

$$M \begin{bmatrix} 40\,000 \\ 60\,000 \end{bmatrix} = 10^3 \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 40 \\ 60 \end{bmatrix} = 10^3 \begin{bmatrix} 32 + 18 \\ 8 + 42 \end{bmatrix} = 10^3 \begin{bmatrix} 50 \\ 50 \end{bmatrix},$$

i.e. L and F have both 50 000 customers

[2] c) What is the expected number of customers each company has on July 1, 2006?

$$M^2 \begin{bmatrix} 40\,000 \\ 60\,000 \end{bmatrix} = M \begin{bmatrix} 50\,000 \\ 50\,000 \end{bmatrix} = 10^3 \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 50 \\ 50 \end{bmatrix} =$$

$$= 10^3 \begin{bmatrix} 40 + 15 \\ 10 + 35 \end{bmatrix} = 10^3 \begin{bmatrix} 55 \\ 45 \end{bmatrix},$$

i.e. L has 55 000 and F has 45 000 customers

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[2] d) Assuming the customer migration matrix M remains the same for a long period, find the percentages of customers using LocoTel and FouPhone in the long-term. That is, solve $MX = X$ to determine the long-term or stable customer distribution X, remembering that the entries of X must add to $1 = 100\%$.

$MX = X \iff (M - I)X = 0$, where

$$M - I = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix};$$

i.e. $2x = 3y$, or $X = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Thus, the long-term customer distribution is $\begin{bmatrix} 60\,000 \\ 40\,000 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & -3 & -2 \\ 1 & -2 & 0 \end{bmatrix}$.

[2] a) Compute the characteristic polynomial to show that the eigenvalues of A are -5 and 1.

$$\det(A-\lambda I_3) = \begin{vmatrix} -\lambda & 2 & 1 \\ 2 & -3-\lambda & -2 \\ 1 & -2 & -\lambda \end{vmatrix} \underset{R_1 \rightsquigarrow R_1+R_3}{=} \begin{vmatrix} -\lambda+1 & 0 & -\lambda+1 \\ 2 & -\lambda-3 & -2 \\ 1 & -2 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -\lambda-3 & -2 \\ 1 & -2 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda-3 & -2 \\ -2 & -\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} 2 & -\lambda-3 \\ 1 & -2 \end{vmatrix} = (1-\lambda)\left[\lambda(\lambda+3) - 4 - 4 + (\lambda+3)\right]$$

$$= (1-\lambda)(\lambda^2+3\lambda+\lambda-4-4+3) = (1-\lambda)(\lambda^2+4\lambda-5)$$

$= (1-\lambda)(\lambda-1)(\lambda+5)$. The zeros of this polynomial are 1 and -5; they are the eigenvalues of A.

[1] b) Find a basis of $E_{-5} = \{v \in \mathbb{R}^3 \mid Av = -5v\}$.

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 2 & 2 & -2 \\ 5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 6 & -12 \\ 0 & 12 & -24 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, i.e. $\begin{matrix} x+z=0 \\ y=2z \end{matrix}$; general solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow$ a basis of E_{-5} is $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

[1] c) Find a basis of $E_1 = \{v \in \mathbb{R}^3 \mid Av = v\}$.

$\begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \\ 1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, i.e. $x-2y-z=0$, general sol is $\begin{bmatrix} 2y+z \\ y \\ z \end{bmatrix} =$

$= y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; a basis of E_1 is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

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[2] d) Find an invertible matrix P, and a diagonal matrix D such that $P^{-1}AP = D$.

$$P = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[1] e) Check that your choice of P in part (d) is invertible.

$$\det P = \begin{vmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \underset{3^{rd}\text{ col}}{=} \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -2-4 = -6 \neq 0,$$

hence P is invertible.

[1] f) (bonus) Find $A^{100}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. We write $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ as a linear combination of eigenvectors:

$$\begin{bmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in E_1.$$

$$\Rightarrow A^{100}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 1^{100}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

15. Consider the matrix $A = \begin{bmatrix} 2 & 4 & -2 & 1 & -1 \\ -2 & -5 & 7 & 3 & 10 \\ 3 & 7 & -8 & 6 & -2 \end{bmatrix}$.

[2] a) Show that the reduced row echelon form of A is

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 9 & 0 & 9 \\ 0 & 1 & -5 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 & -1 \\ -2 & -5 & 7 & 3 & 10 \\ 3 & 7 & -8 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1/2 & -1/2 \\ 0 & -1 & 5 & 4 & 9 \\ 3 & 7 & -8 & 6 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1/2 & 1/2 \\ 0 & -1 & 5 & 4 & 9 \\ 0 & 1 & -5 & 4\tfrac{1}{2} & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1/2 & -1/2 \\ 0 & 1 & -5 & -4 & -9 \\ 0 & 0 & 0 & 8\tfrac{1}{2} & 8\tfrac{1}{2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1/2 & -1/2 \\ 0 & 1 & -5 & -4 & -9 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 0 & 1 & -5 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \tilde{A}$$

[3] b) Give a basis for the null space Nul A of the matrix A.

$\text{Nul } A = \{ X \in \mathbb{R}^5 : AX = 0 \} = \{ X \in \mathbb{R}^5 : \tilde{A} X = 0 \}$,

general solution
$$x_1 = -9x_3 - 9x_5$$
$$x_2 = 5x_3 + 5x_5$$
$$x_4 = -x_5$$

i.e. $\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} -9x_3 - 9x_5 \\ 5x_3 + 5x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9 \\ 5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

a basis of Nul A is $\left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 5 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

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[1] c) What is the dimension of ker A?

dim Ker A = dim Nul A = # of basis elements in Nul A
= 2

[1] d) Give a basis for the column space of A.

A basis of col A is given by the pivot columns of A, ie columns 1, 2, 4.

Thus a basis of col A is $\left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$.

[1] e) Do the columns of A span $\mathbb{R}^3$? Yes, since dim Col(A) = 3 = dim $\mathbb{R}^3$,

we have Col(A) = $\mathbb{R}^3$.