



## MAT 1302A Final Exam

April 27, 2005      Duration: 3 hours.

Instructor: Barry Jessup.

Family Name: \_\_\_\_\_

First Name: \_\_\_\_\_

Student number: \_\_\_\_\_

**PLEASE READ THESE INSTRUCTIONS CAREFULLY.**

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted**.
3. Read each question carefully -you will save yourself time and unnecessary grief later on.
4. Questions 1 to 10 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the space opposite.
5. Questions 11 –15 require a complete solution. The number of points each part is worth is indicated in square brackets, so spend your time accordingly. Answer these questions in the space provided, and use the backs of pages if necessary.
6. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

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Total	

1. For a nonhomogeneous system of 10 equations in 12 unknowns, answer the following three questions:

- Can the system be inconsistent?
- Can the system have infinitely many solutions?
- Can the system have only one solution?

- A. Yes, Yes, No.
- B. No, No, Yes.
- C. Yes, No, Yes.
- D. No, Yes, Yes.
- E. Yes, Yes, Yes.
- F. No, No, No.

2. Find all values of  $a$  and  $b$  so that the following system is inconsistent.

$$\begin{array}{rccccccc} x & + & y & + & 3z & = & 2 \\ & & y & + & 2z & = & -1 \\ & & & & (a - 6)z & = & b - 4 \end{array}$$

- A.  $a \neq 6$ ,  $b \neq 4$
- B.  $a = 6$ ,  $b \neq 4$
- C.  $a \neq 6$ ,  $b = 4$
- D.  $a = 6$ ,  $b = 4$
- E.  $a$  is arbitrary,  $b \neq 4$
- F.  $a = 6$ ,  $b$  is arbitrary

3. If  $P = \begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$ , which of the following is the product  $QP$ ?

A.  $\begin{bmatrix} 3 & -7 & 1 \\ 6 & -9 & 0 \\ 6 & -4 & -2 \end{bmatrix}$

B.  $\begin{bmatrix} 3 & -7 & 3 \\ 3 & -10 & 1 \\ 0 & -12 & 4 \end{bmatrix}$

C.  $\begin{bmatrix} 3 & 3 & 0 \\ -7 & -12 & -10 \\ 1 & 3 & 4 \end{bmatrix}$

D.  $\begin{bmatrix} -3 & 7 & -1 \\ -6 & 9 & 0 \\ -6 & 4 & 2 \end{bmatrix}$

E.  $\begin{bmatrix} 3 & -4 & 1 \\ -10 & -7 & -12 \\ 3 & 0 & 3 \end{bmatrix}$

F.  $\begin{bmatrix} -3 & -6 & -6 \\ 7 & 9 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

4. Compute the determinant  $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$ .

A. -6

B. 12

C. -12

D. -18

E. 18

F. -24

5. Suppose  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ . Which one of the following statements is true ?

- A.  $A^{-1}$  does not exist.
- B. The third row of  $A^{-1}$  is  $[-1 \ -1 \ 1]$ .
- C. The second row of  $A^{-1}$  is  $[1 \ 2 \ -1]$ .
- D. The first row of  $A^{-1}$  is  $[2 \ 0 \ -1]$ .
- E. The second column of  $A^{-1}$  is  $[0 \ 2 \ -1]^t$ .
- F. The first column of  $A^{-1}$  is  $[1 \ 2 \ 1]^t$ .

6. Compute  $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2005}$ . (Hint: Use the partition  $\left[ \begin{array}{cc|cc} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$ )

A.  $\begin{bmatrix} 1 & 0 & 0 & 2005 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B.  $\begin{bmatrix} 1 & 0 & 0 & 6015 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C.  $\begin{bmatrix} 1 & 0 & 0 & -3^{2005} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D.  $\begin{bmatrix} 1 & 0 & 0 & 2005^{-3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E.  $\begin{bmatrix} 1 & 0 & 0 & -2005 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F.  $\begin{bmatrix} 1 & 0 & 0 & -6015 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7. Find all  $x$  in  $\mathbf{R}$  such that  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ x \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$  is a linearly independent set.

- A. All  $x$  except 0.
- B. All  $x$  except 0 and 1.
- C. All  $x$  except 2.
- D.  $x = 0, 1, 2$ .
- E. All  $x$  except 3.
- F.  $x = 3$ .

8. Find the value of  $t$  for which  $\begin{bmatrix} 4 \\ -3 \\ 5 \\ t \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 1 \\ 3 \\ -3 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ -1 \\ -4 \\ 3 \end{bmatrix}$ .

- A. -4
- B. 0
- C. 3
- D. 7
- E. 9
- F. 11

9. Which of the following are subspaces of  $\mathbf{R}^3$ ?

$$U = \left\{ \begin{bmatrix} x \\ y \\ x - y \end{bmatrix} \mid x, y \in \mathbf{R} \right\}$$

$$V = \left\{ \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} \mid x, y \in \mathbf{R} \right\}$$

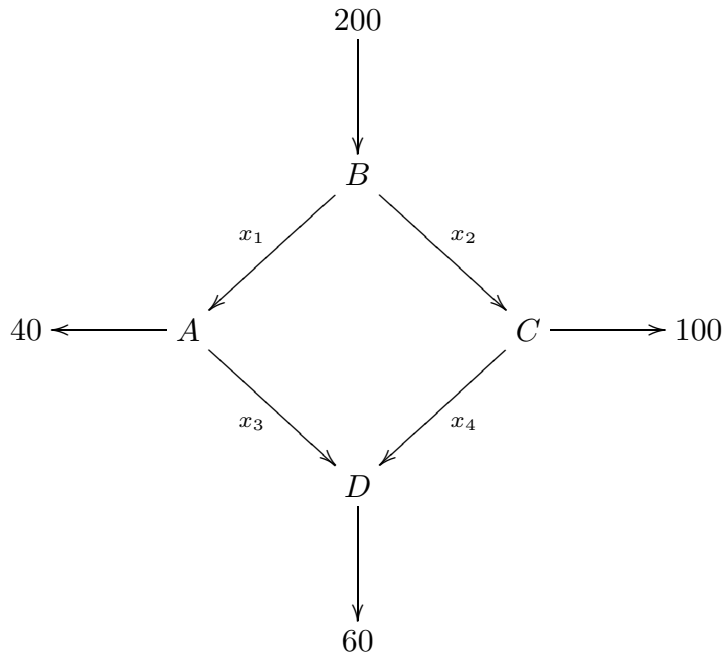
$$W = \left\{ \begin{bmatrix} x \\ y \\ xy \end{bmatrix} \mid x, y \in \mathbf{R} \right\}$$

- A.  $U$  and  $V$  only
- B.  $U$  and  $W$  only
- C.  $V$  and  $W$  only
- D.  $U$  only
- E.  $V$  only
- F.  $W$  only

10. Suppose  $M$  is an invertible  $n \times n$  matrix. Which of the following statements is false?

- A. The columns of  $M$  are linearly independent.
- B.  $Mx = 0$  has a non-trivial solution.
- C.  $Mx = b$  has a unique solution for every  $b \in \mathbf{R}^n$ .
- D. The determinant of  $M$  is not zero.
- E.  $M$  is row-equivalent to the identity matrix.
- F. The rank of  $M$  is  $n$ .

11. Consider the network of streets with intersections A, B, C, and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter or leave A, B, C, and D during one minute. Each  $x_i$  denotes the unknown number of cars which passed along the indicated streets during the same period.



[3] a) Write down the linear system which describes the traffic flow, **together with all the constraints** on the variables  $x_i$ ,  $i = 1, \dots, 4$ . (*Do not perform any operations on your equations: this is done for you in (b)*)

[2] b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Give the general solution. (Ignore the constraints at this point.)

[1] c) Using (b) and your constraints from (a), find the largest flow along BC.

[1] d) If the flow along CD is limited to at most 40 cars per minute because of roadwork, what is the largest possible flow along BC?





**12.** An open economy has 2 sectors, Manufacturing and Services. To produce one unit of output, the Manufacturing sector requires .25 units of Manufacturing and .2 units of Services, while the Services sector requires .5 units of Manufacturing and .2 units of Services to produce one unit of its own output.

- [2] a) Give the the input-output or consumption matrix  $C$  for this economy.
- [2] b) Give the Leontief Input-Output Model equation, relating the production vector  $x$ , the final demand vector  $d$  and the input-output or consumption matrix  $C$ .
- [2] c) Find the inverse of  $(I - C)$ .
- [2] d) What is the production vector necessary to satisfy a final demand vector  $d = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ . Use the inverse of  $(I - C)$  in your calculation.



**13.** Two long-distance telephone companies A and B compete for customers. Market research has shown that in any 3-month period, 80% of A's customers stay with company A, while 20% switch to company B. In the same period, 30% of B's customers switch to company A, while 70% stay with company B. The following table summarizes these customer 'migrations'.

A	B	
.8	.3	A
.2	.7	B

On January 1, 2005, company A has 5 million customers and company B has 10 million, i.e. the initial customer vector is  $X_0 = 10^6 \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ . Assume that the total number of customers does not increase during a 12 month period.

- [1] a) Give the customer 'migration' matrix  $M$ .
- [2] b) What is the expected number of customers each company has on April 1, 2005? (i.e. just one 3-month period after January 1st.)
- [2] c) What is the expected number of customers each company has on July 1, 2005?
- [2] d) Assuming the customer migration matrix  $M$  remains the same for a long period, find the percentages of customers using company A and company B in the long-term. That is, solve  $MX = X$  to determine the long-term or stable customer distribution  $X$  (also known as the steady-state vector), remembering that the entries of  $X$  must add to  $1 = 100\%$ .



**14.** Let  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ .

- [1] a) Compute the characteristic polynomial to show that the eigenvalues of  $A$  are  $-1$  and  $5$ .
- [2] b) Find a basis of  $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$ .
- [2] c) Find a basis of  $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$ .
- [2] d) Find an invertible matrix  $P$ , and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .
- [1] e) Check that your choice of  $P$  in part (d) is invertible.



15. Consider the matrix  $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 1 \\ 3 & -6 & 9 & 1 \end{bmatrix}$ , whose reduced row echelon form is

$$\tilde{A} = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] a) Give a basis for the column space  $\text{col}(A)$ .

[3] b) Give a basis for the kernel (or nullspace)  $\ker A$ .

[1] c) What is the dimension of  $\ker A$ ?

[1] d) What is the rank of  $A$ ?

[1] e) Are the columns of  $A$  linearly independent ?

