1. For a nonhomogeneous system of 10 equations in 12 unknowns, answer the following three questions:

$$\begin{array}{cc} & \overbrace{}^{12} \\ 10 & [A \mid b] \end{array} \qquad b \neq 0$$

- Can the system be inconsistent? Yes
- Can the system have infinitely many solutions? Yes
- Can the system have only one solution? No

rank $A \leq 10$

$\therefore$ if consistent, # parameters $\geq 12 - 10 = 2$

A. Yes, Yes, No.
B. No, No, Yes.
C. Yes, No, Yes.
D. No, Yes, Yes.
E. Yes, Yes, Yes.
F. No, No, No.

2. Find all values of a and b so that the following system is inconsistent.

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 2 \\ & & y & + & 2z & = & -1 \\ & & & & (a-6)z & = & b-4 \end{array}$$

A. $a \neq 6,\ b \neq 4$
B. $a = 6,\ b \neq 4$
C. $a \neq 6,\ b = 4$
D. $a = 6,\ b = 4$
E. a is arbitrary, $b \neq 4$
F. $a = 6,\ b$ is arbitrary

if $a = 6$ and $b \neq 4$,

have $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & * \end{array}\right]$, when $* \neq 0$;

this is inconsistent; moreover, it is consistent in all other cases.

3. If $P = \begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$, which of the following is the product QP?

A. $\cancel{\begin{bmatrix} 3 & -7 & 1 \\ 6 & -9 & 0 \\ 6 & -4 & -2 \end{bmatrix}}$ B. $\cancel{\begin{bmatrix} 3 & -7 & 3 \\ 3 & -10 & 1 \\ 0 & -12 & 4 \end{bmatrix}}$ C. $\begin{bmatrix} 3 & 3 & 0 \\ -7 & -12 & -10 \\ 1 & 3 & 4 \end{bmatrix}$

D. $\begin{bmatrix} -3 & 7 & -1 \\ -6 & 9 & 0 \\ -6 & 4 & 2 \end{bmatrix}$ E. $\cancel{\begin{bmatrix} 3 & -4 & 1 \\ -10 & -7 & -12 \\ 3 & 0 & 3 \end{bmatrix}}$ F. $\cancel{\begin{bmatrix} -3 & -6 & -6 \\ 7 & 9 & 4 \\ -1 & 0 & 2 \end{bmatrix}}$

$$QP = \begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 7 & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

(there's no need to go any further..

4. Compute the determinant $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$. $= 2 \cdot 1 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

$$= 2 \cdot 1 \cdot 3 \cdot \underbrace{\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}}_{-3}$$

$$= -18$$

A. -6
B. 12
C. -12
D. -18
E. 18
F. -24

5. Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.

B. The third row of A^{-1} is $[-1 \quad -1 \quad 1]$.

C. The second row of A^{-1} is $[1 \quad 2 \quad -1]$.

D. The first row of A^{-1} is $[2 \quad 0 \quad -1]$.

E. The second column of A^{-1} is $[0 \quad 2 \quad -1]^t$.

F. ~~All of B, C, D and E are true.~~

The first column of A^{-1} is $[1\ 2\ 1]^t$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} & * & & & * & \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

6. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2005}$. (Hint: Use the partition $\left[\begin{array}{cc|cc} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$)

A. $\begin{bmatrix} 1 & 0 & 0 & 2005 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 6015 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & -3^{2005} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 2005^{-3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & -2005 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 0 & -6015 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2N \\ 0 & 1 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2N \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3N \\ 0 & 1 \end{bmatrix}$$

$$\therefore M^k = \begin{bmatrix} 1 & kN \\ 0 & 1 \end{bmatrix}$$

$$\therefore M^{2005} = \begin{bmatrix} 1 & 2005 \cdot N \\ 0 & 1 \end{bmatrix}$$

7. Find all x in $\mathbf{R}$ such that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ x \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set.

A. All x except 0.

B. All x except 0 and 1.

C. All x except 2.

D. $x = 0, 1, 2$.

E. All x except 3.

F. $x = 3$.

$$\det \begin{bmatrix} 1 & -2 & 2 \\ 1 & x & -1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ 0 & x+2 & -3 \\ 0 & 5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & -3 \\ 5 & -3 \end{vmatrix} = -3x - 6 + 15 = -3x + 9 \neq 0$$

$$\Leftrightarrow x \neq 3$$

8. Find the value of t for which $\begin{bmatrix} 4 \\ -3 \\ 5 \\ t \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 3 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -4 \\ 3 \end{bmatrix}$.

A. -4

B. 0

C. 3

D. 7

E. 9

F. 11

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & -4 & 5 \\ 0 & -3 & 3 & t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & 14 \\ 0 & 0 & 0 & t-9 \end{array}\right]$$

This is consistent $\Leftrightarrow$ $t = 9$.

9. Which of the following are subspaces of $\mathbf{R}^3$?

$$U = \{ \begin{bmatrix} x \\ y \\ x-y \end{bmatrix} \mid x, y \in \mathbf{R} \} = \{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \mid x, y \in \mathbb{R} \}$$

$$= \text{span} \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \therefore U \text{ is a s.s.}$$

$$V = \{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \mid x, y \in \mathbf{R} \} = \text{Span} \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \} \therefore V \text{ is a s.s.}$$

$$W = \{ \begin{bmatrix} x \\ y \\ xy \end{bmatrix} \mid x, y \in \mathbf{R} \}$$

(So... the answer must be "A", since "U, V, W" is not an option)

A. U and V only

B. U and W only

C. V and W only

D. U only

E. V only

F. W only

Indeed, W is not closed under addition:

e.g. $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in W$ and $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in W$ but

$u + v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin W$ (since the 3rd component should be $1 \cdot 1 = 1$)

10. Suppose M is an <u>invertible</u> $n \times n$ matrix. Which of the following statements is false?

A. The columns of M are linearly independent. ✓

B. $Mx = 0$ has a non-trivial solution.

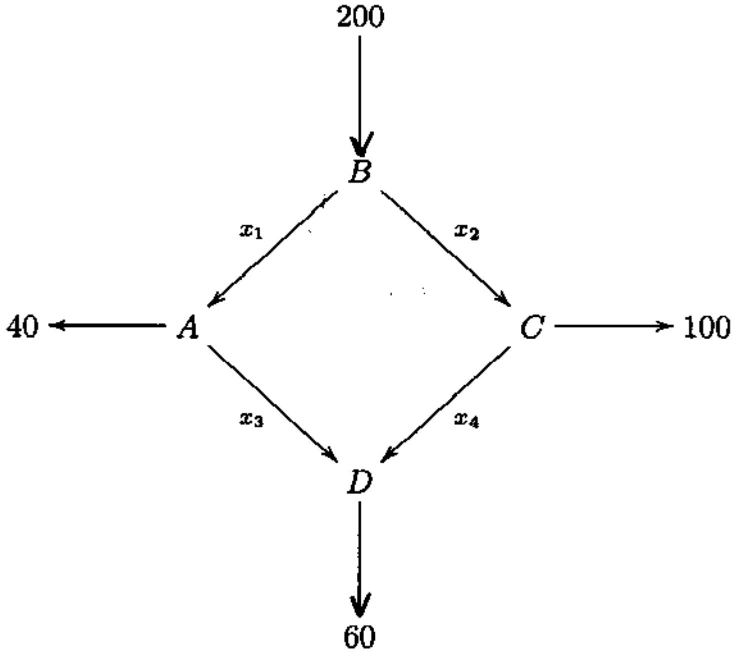
C. $Mx = b$ has a unique solution for every $b \in \mathbf{R}^n$. ✓

D. The determinant of M is not zero. ✓

E. M is row-equivalent to the identity matrix. ✓

F. The rank of M is n. ✓

11. Consider the network of streets with intersections A, B, C, and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter or leave A, B, C, and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



[3] a) Write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 4$. *(Do not perform any operations on your equations: this is done for you in (b))*

[2] b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \begin{array}{cccc|c} & & & s & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints at this point.)

[1] c) Using (b) and your constraints from (a), find the largest flow along BC.

[1] d) If the flow along CD is limited to at most 40 cars per minute because of roadwork, what is the largest possible flow along BC?

a)

	In	=	Out
A	x_1	=	$x_3 + 40$
B	200	=	$x_1 + x_2$
C	x_2	=	$x_4 + 100$
D	$x_3 + x_4$	=	60

$x_i \geq 0$ (one-way streets)

$x_i \in \mathbb{Z}$ (#'s of cars)

b) $x_1 = 100 - \lambda$
$x_2 = 100 + \lambda$
$x_3 = 60 - \lambda$; $\lambda \in \mathbb{R}$
$x_4 = \lambda$

c) $x_i \geq 0 \Leftrightarrow \lambda \geq 0$; $60 - \lambda \geq 0$ i.e. $60 \geq \lambda$

$100 + \lambda \geq 0$ (OK) ; $100 - \lambda \geq 0 \Leftrightarrow \lambda \leq 100$

$\therefore \quad 0 \leq \lambda \leq 60$

$\therefore$ largest flow along BC = largest value of $x_2 = 100 + \lambda$
$= 100 + 60$
$= 160$

d) CD's flow $\leq 40 \Leftrightarrow x_4 = \lambda \leq 40$. So,
Flow along BC $x_2 = 100 + \lambda \leq 140$ $\therefore$ largest flow along BC is 140

12. An open economy has 2 sectors, Manufacturing and Services. To produce one unit of output, the Manufacturing sector requires .25 units of Manufacturing and .2 units of Services, while the the Services sector requires .5 units of Manufacturing and .2 units of Services to produce one unit of its own output.

[2] a) Give the the input-output or consumption matrix C for this economy.

[2] b) Give the Leontief Input-Output Model equation, relating the production vector x, the final demand vector d and the input-output or consumption matrix C.

[2] c) Find the inverse of $(I - C)$.

[2] d) What is the production vector necessary to satisfy a final demand vector $d = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. Use the inverse of $(I - C)$ in your calculation.

a) $C = \begin{bmatrix} .25 & .5 \\ .2 & .2 \end{bmatrix}$

b) $x = Cx + d$

c) $I - C = \begin{bmatrix} .75 & -.5 \\ -.2 & .8 \end{bmatrix} = \begin{bmatrix} 3/4 & -\frac{1}{2} \\ -1/5 & 4/5 \end{bmatrix} \therefore (I-C)^{-1} = \dfrac{1}{\frac{12}{20} - \frac{1}{10}} \begin{bmatrix} 4/5 & \frac{1}{2} \\ \frac{1}{5} & 3/4 \end{bmatrix}$

$= 2 \cdot \begin{bmatrix} 4/5 & \frac{1}{2} \\ \frac{1}{5} & 3/4 \end{bmatrix} = \begin{bmatrix} 8/5 & 1 \\ 2/5 & 3/2 \end{bmatrix}$

d) By (b), $d = (I - C)x$, so $x = (I-C)^{-1} d$

$= \begin{bmatrix} 8/5 & 1 \\ 2/5 & 3/2 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 360 \\ 340 \end{bmatrix}$

13. Two long-distance telephone companies A and B compete for customers. Market research has shown that in any <u>3-month period</u>, 80% of A's customers stay with company A, while 20% switch to company B. In the same period, 30% of B's customers switch to company A, while 70% stay with company B. The following following table summarizes these customer 'migrations'.

A	B	
.8	.3	A
.2	.7	B

On January 1, 2005, company A has 5 million customers and company B has 10 million, i.e. the initial customer vector is $X_0 = 10^6 \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

[1] a) Give the customer 'migration' matrix M.

[2] b) What is the expected number of customers each company has on April 1, 2005? (i.e. just <u>one</u> 3-month period after January 1st.)

[2] c) What is the expected number of customers each company has on July 1, 2005?

[2] d) Assuming the customer migration matrix M remains the same for a long period, find the percentages of customers using company A and company B in the long-term. That is, solve $MX = X$ to determine the long-term or stable customer distribution X (also known as the steady-state vector), remembering that the entries of X must add to $1 = 100\%$.

a) $M = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$

b) $X_1 = MX_0 = 10^6 \times \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 10^6 \times \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$\therefore$ company A will have 7×10^6 customers
and " B " 8×10^6 " .

c) July 1, 2005 is 1 3 month period after April 1, so $X_2 = MX_1 = 10^6 \times \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$= 10^6 \times \begin{bmatrix} 5.6 + 2.4 \\ 1.4 + 5.6 \end{bmatrix} = 10^6 \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ $\therefore$ company A will have 8×10^6 customers
& " B " 7×10^6 " .

d) We solve $MX = X$; or $(M - I)x = 0$: $[M - I \mid 0] = \left[\begin{array}{cc|c} -.2 & .3 & 0 \\ .2 & -.3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

d) cont'd. $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2\, s \\ s \end{bmatrix}$, $s \in \mathbb{R}$.

But we need $\frac{3}{2}s + s = 1$, so $\frac{5}{2}s = 1$ i.e. $s = \frac{2}{5}$

$$\therefore X = \begin{bmatrix} \frac{3}{2} \cdot \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

i.e. in the long term, 60% of the customers will be using company A &
40% " " B.

14. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

[1] a) Compute the characteristic polynomial to show that the eigenvalues of A are -1 and 5.

[2] b) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

[2] c) Find a basis of $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$.

[2] d) Find an invertible matrix P, and a diagonal matrix D such that $P^{-1}AP = D$.

[1] e) Check that your choice of P in part (d) is invertible.

a) $$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 3-\lambda \\ 2 & 1-\lambda & 4 \\ 1+\lambda & 0 & 0 \end{vmatrix} = (1+\lambda)\begin{vmatrix} 2 & 3-\lambda \\ 1-\lambda & 4 \end{vmatrix}$$

$$= (1+\lambda)\{8 - (1-\lambda)(3-\lambda)\} = (1+\lambda)\{8 - [3 - 4\lambda + \lambda^2]\} = (1+\lambda)\{5 + 4\lambda - \lambda^2\} = (1+\lambda)^2(5-\lambda)$$

$\therefore\ |A-\lambda I| = 0 \iff \lambda = -1, 5.$

b) $E_{-1} = \ker(A+I)$: $$[A+I \mid 0] = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \quad\ \ s \ \ \ t \quad \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{-1}

c) $E_5 = \ker(A-5I)$: $$[A-5I \mid 0] = \left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \qquad\quad s \quad \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ is a basis for E_5

d) Let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

e) $$\det P = \begin{vmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$$ $\therefore$ P is invertible

15. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 1 \\ 3 & -6 & 9 & 1 \end{bmatrix}$, whose reduced row echelon form is

$$\tilde{A} = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] a) Give a basis for the column space col(A).

[3] b) Give a basis for the kernel (or nullspace) ker A.

[1] c) What is the dimension of ker A?

[1] d) What is the rank of A?

[1] e) Are the columns of A linearly independent ?

a) Since leading ones are in cols 1, 4 of $\tilde{A}$, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of col A

b) $[A \mid 0] \sim [\tilde{A} \mid 0] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ with s over the second column and t over the third $\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s-3t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for ker A

c) dim ker A = # of vectors in the basis = 2

d) rank A = # leading ones in $\tilde{A}$ = 2

e) NO, Since – 4 vectors in $\mathbb{R}^3$ are always dependent
OR – cols of A are l.i. $\Leftrightarrow$ rank A = # cols, but
rank A = 2 & # cols = 4.