MAT 1302: MATHEMATICAL METHODS II

Section 1302A: Professor Monica Nevins

Final Exam (April 20, 2004)

Family Name ____________________

First Name: ____________________

Student number: ____________________

Instructions:

- There are 12 questions. The first 4 are short-answer, and have multiple parts. The last 8 are long-answer, and partial credit may be earned. Please be careful to include all details, and explain what you are doing.
- You have 180 minutes. The exam is out of 100 points.
- No books or notes allowed. Only simple calculators are permitted; none with graphing or matrix capabilities.
- If you do not have enough space, use the back of the pages and clearly indicate this. The exam has 13 pages.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
maximal	8	6	8	6	8	6	8	12	8	10	8	12	100
your score													

1. (8 points) Suppose A is a 5×7 matrix and that the dimension of the null space of A is 2. Answer the following questions in the table provided; only this table will be marked; 1 point for every correct answer.

(a) What is the rank of A?

(b) Are the columns of A linearly dependent?

(c) Do the nonpivot columns of A form a basis for the null space of A?

(d) Is there necessarily a row of zeros in a RREF (= reduced row echelon form) of A?

(e) Is there necessarily a column of zeros in a RREF of A?

(f) The null space of A is a subspace of $\mathbb{R}^p$ for which value of p?

(g) Can the linear system $A\mathbf{x} = \mathbf{b}$ be inconsistent for some choice of $\mathbf{b}$?

(h) Will the linear system $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)	(**g**)	(**h**)
your answer								

2. (6 points) Suppose A is a 6×6 **invertible** matrix. Answer the following questions in the table provided; only this table will be marked; 1 point for every correct answer.

(a) What is the rank of A?

(b) What is $\dim(\text{Nul}\,(A))$?

(c) Can the linear system $A\mathbf{x} = \mathbf{b}$ be inconsistent for some choice of $\mathbf{b}$?

(d) Is $\det(A) = 0$?

(e) How many vectors are there in a basis for $\text{Col}\,(A)$?

(f) Is A necessarily diagonalizable?

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)
your answer						

3. (8 points) Answer each of the following short-answer questions in the table provided. **Only this table will be graded. 2 points for each correct answer; no partial credit.**

(a) Compute the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \\ -4 & 1 & 1 \end{bmatrix}$$

(b) Find the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

(c) If $z = 6 + 4i$ and $w = 1 - i$, compute $z\overline{w}$.

(d) If $z = 6 + 4i$ and $w = 1 - i$, compute $\frac{z}{w}$.

Problem	(a)	(b)	(c)	(d)
your answer				

4. (6 points) Answer each of the following short-answer questions. **No partial credit.**

(a) (2 points) Is the set of all $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ satisfying $2x - 5y = 0$ a subspace? Answer yes or no.

Your answer: ________________________

(b) (2 points) Compute the product, if possible:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} =$$

(c) (2 points) Define a basis of a subspace W of $\mathbb{R}^n$.

5. (8 points) Let α denote the last digit of your student number. Given the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ \alpha \end{bmatrix}.$$

(a) (6 points) Find an equation in the variables b_1, b_2, b_3 which describes the set of all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) (2 points) Give one vector $\mathbf{u}$ in $\mathbb{R}^3$ which is NOT in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. (Hint: use part (a).)

6. (6 points) Given the following matrix A

$$A = \begin{bmatrix} 2 & 4 & -5 & -2 & -16 \\ 3 & 6 & -2 & -3 & -13 \\ -2 & -4 & 4 & 2 & 14 \end{bmatrix}$$

whose reduced row echelon form is

$$R = \begin{bmatrix} 1 & 2 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (1 points) Give a basis for $\mathrm{Col}\,(A)$:

(b) (3 points) Give a basis for $\mathrm{Nul}\,(A)$.

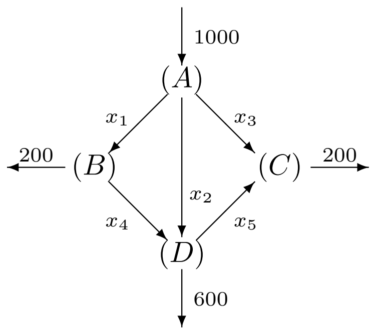
(c) (1 points) What is the rank of A?

Your answer: ______________________

(d) (1 points) What is $\dim(\mathrm{Nul}\,(A))$?

Your answer: ______________________

7. (8 points) The internet traffic flow among four major hubs is described in the diagram below. The quantity of traffic is measured in megabits per second, or mbps. The numbers in the diagram represent the traffic flow into the hubs on a typical day. **Traffic may only flow in the directions indicated.**



(a) (2 points) Give a system of linear equations governing the flow of internet traffic in this network.

(b) (4 points) Solve the linear system in (a) using row reduction.

(c) (2 points) Suppose the wire marked x_3 is cut, and can no longer accept traffic, i.e., $x_3 = 0$. What is the resulting maximum internet traffic flow on x_2?

8. (a) (10 points) Find the inverse of the following matrix A and write your answer in the space provided. You must show your work and explain what you are doing to get full marks. (Hint: The inverse will be a matrix with integer entries.)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \qquad A^{-1} =$$

(b) (2 points) Check your answer carefully by multiplying AA^{-1}.

9. (8 points) (Leontief input-output model) Two companies (A) and (B) use each others services for their production. For each \$1 of output, (A) uses $\$\frac{1}{4}$ of its own products and $\$\frac{1}{2}$ worth of products of (B), while (B) uses for each \$1 of output $\$\frac{2}{5}$ worth of its own products and $\$\frac{1}{2}$ worth of products of (A).

(a) (2 points) Give the consumption matrix of this economy.

(b) (6 points) What is the total production output $\mathbf{x}$ needed to meet a final demand of $\mathbf{d} = \begin{bmatrix} \$1,000 \\ \$2,000 \end{bmatrix}$? What is the intermediate demand in this case?

10. (10 points) The matrix $A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & 8 \\ -1 & -2 & 6 \end{bmatrix}$ has exactly two eigenvalues, 1 and 2.

(a) (4 points) Find a basis for the eigenspace corresponding to the eigenvalue 1.

(b) (4 points) Find a basis for the eigenspace corresponding to the eigenvalue 2.

(c) (2 points) Is A diagonalizable? If so, give a diagonalization of A, that is, give matrices P and D such that P is an invertible matrix, D is a diagonal matrix and $A = PDP^{-1}$; if not, explain why not, referring to any theorems discussed in class.

11. (a) (3 points) Give the definition of p vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ to be linearly independent.

(b) (5 points) Let k be a real number. Determine for which value or values of k the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} k \\ 3 \\ 0 \end{bmatrix}$$

are linearly independent.

12. (12 points) Suppose the matrix

$$M = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

describes the dynamics of some system, e.g., M is the migration matrix describing the flow of customers between two companies.

(a) (2 points) Check that $\lambda = 1$ and $\lambda = 0.92$ are eigenvalues for that system.

(b) (4 points) Find the corresponding eigenvectors.

(c) (4 points) Represent $\mathbf{x} = \begin{bmatrix} 1/10 \\ 3/2 \end{bmatrix}$ as a linear combination of eigenvectors.

(d) (2 points) Find $M^2\mathbf{x}$. (Hint: use (c).)