MAT 1302: MATHEMATICAL METHODS II

Section 1302 B: Professor Erhard Neher

Final Exam (April 20, 2004)

Family Name ______________________________

First Name: ______________________________

Student number: __________________________

Instructions:

- There are 12 questions. The first 4 are short-answer, and have multiple parts. The last 8 are long-answer, and partial credit may be earned. Please be careful to include all details, and explain what you are doing.
- You have 180 minutes. The exam is out of 100 points.
- No books or notes allowed. Only simple calculators are permitted; none with graphing or matrix capabilities.
- If you do not have enough space, use the back of the pages and clearly indicate this. The exam has 13 pages.

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
maximal	8	6	8	6	8	6	8	12	8	10	8	12	100
your score													

1. (8 points) Suppose A is a 5×7 matrix and that the dimension of the null space of A is 2. Answer the following questions in the table provided; only this table will be marked; 1 point for every correct answer.

(a) What is the rank of A?

(b) Are the columns of A linearly dependent?

(c) Do the nonpivot columns of A form a basis for the null space of A?

(d) Is there necessarily a row of zeros in a RREF (= reduced row echelon form) of A?

(e) Is there necessarily a column of zeros in a RREF of A?

(f) The null space of A is a subspace of $\mathbb{R}^p$ for which value of p?

(g) Can the linear system $A\mathbf{x} = \mathbf{b}$ be inconsistent for some choice of $\mathbf{b}$?

(h) Will the linear system $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)	(**g**)	(**h**)
your answer								

Answer:

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)	(**g**)	(**h**)
your answer	5	yes	no	no	no	7	no	no

2. (6 points) Suppose A is a 6×6 **invertible** matrix. Answer the following questions in the table provided; only this table will be marked; 1 point for every correct answer.

(a) What is the rank of A?

(b) What is $\dim(\text{Nul}\,(A))$?

(c) Can the linear system $A\mathbf{x} = \mathbf{b}$ be inconsistent for some choice of $\mathbf{b}$?

(d) Is $\det(A) = 0$?

(e) How many vectors are there in a basis for $\text{Col}\,(A)$?

(f) Is A necessarily diagonalizable?

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)
your answer						

Answer:

Problem	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)	(**f**)
your answer	6	0	no	no	6	no

3. (8 points) Answer each of the following short-answer questions in the table provided. **Only this table will be graded. 2 points for each correct answer; no partial credit.**

(a) Compute the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \\ -4 & 1 & 1 \end{bmatrix}$$

(b) Find the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

(c) If $z = 6 + 4i$ and $w = 1 - i$, compute $z\overline{w}$.

(d) If $z = 6 + 4i$ and $w = 1 - i$, compute $\frac{z}{w}$.

Problem	**(a)**	**(b)**	**(c)**	**(d)**
your answer				

Answer:

Problem	**(a)**	**(b)**	**(c)**	**(d)**
your answer	-14	$3 \pm 4i$	$2 + 10i$	$1 + 5i$

4. (6 points) Answer each of the following short-answer questions. **No partial credit.**

(a) (2 points) Is the set of all $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ satisfying $2x - 5y = 0$ a subspace? Answer yes or no.

Your answer: ______________________

Answer: Yes.

(b) (2 points) Compute the product, if possible:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} =$$

Answer: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$.

(c) (2 points) Define a basis of a subspace W of $\mathbb{R}^n$.

Answer: A basis is a subset of W which is linearly independent and spans W.

5. (8 points) Let α denote the last digit of your student number. Given the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ \alpha \end{bmatrix}.$$

(a) (6 points) Find an equation in the variables b_1, b_2, b_3 which describes the set of all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) (2 points) Give one vector $\mathbf{u}$ in $\mathbb{R}^3$ which is NOT in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. (Hint: use part (a).)

Answer: (a) A vector $\mathbf{b}$ lies in the span if it is a liner combination of the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. This means that the linear system $A\mathbf{x} = \mathbf{b}$ is solvable, where A is the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{bmatrix} 1 & -1 & | & b_1 \\ 2 & -1 & | & b_2 \\ 3 & \alpha & | & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & | & b_1 \\ 0 & 1 & | & -2b_1 + b_2 \\ 0 & 3+\alpha & | & -3b_1 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & | & b_1 \\ 0 & 1 & | & -2b_1 + b_2 \\ 0 & 0 & | & (3+2\alpha)b_1 - (3+\alpha)b_2 + b_3 \end{bmatrix}$$

So we need $(3+2\alpha)b_1 - (3+\alpha)b_2 + b_3 = 0$. (b) For example, $(0,0,1)$ does not lie in the span since its coordinates do not fulfill the equation (a).

6. (6 points) Given the following matrix A

$$A = \begin{bmatrix} 2 & 4 & -5 & -2 & -16 \\ 3 & 6 & -2 & -3 & -13 \\ -2 & -4 & 4 & 2 & 14 \end{bmatrix}$$

whose reduced row echelon form is

$$R = \begin{bmatrix} 1 & 2 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (1 points) Give a basis for $\text{Col}\,(A)$:

(b) (3 points) Give a basis for $\text{Nul}\,(A)$.

(c) (1 points) What is the rank of A?

Your answer: ____________________

(d) (1 points) What is $\dim(\text{Nul}\,(A))$?

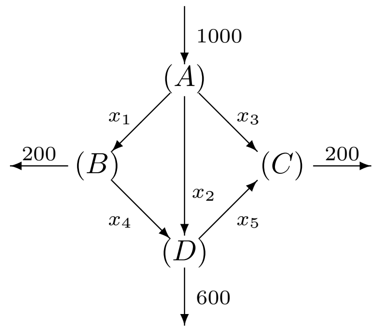
Your answer: ____________________

Answer: (a) Columns 1 and 3 of A. (b) The linear system is equivalent to $x_1 = -2x_2 + x_4 + 3x_5$, $x_3 = -2x_5$ and x_2, x_4, x_5 free variables. Hence a basis is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(c) 2, (d) 3, because of (b) or the Rank Theorem.

7. (8 points) The internet traffic flow among four major hubs is described in the diagram below. The quantity of traffic is measured in megabits per second, or mbps. The numbers in the diagram represent the traffic flow into the hubs on a typical day. **Traffic may only flow in the directions indicated.**



(a) (2 points) Give a system of linear equations governing the flow of internet traffic in this network.

Answer:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1000 \\
x_1 - x_4 &= 200 \\
x_2 + x_4 - x_5 &= 600 \\
x_3 + x_5 &= 200
\end{aligned}
$$

(b) (4 points) Solve the linear system in (a) using row reduction.

Answer: $x_1 = 200 + x_4$, $x_2 = 600 - x_4 + x_5$, $x_3 = 200 - x_5$, x_4, x_5 are free

(c) (2 points) Suppose the wire marked x_3 is cut, and can no longer accept traffic, i.e., $x_3 = 0$. What is the resulting maximum internet traffic flow on x_2?

Answer: From (b) we get $x_5 = 200$, hence $x_2 = 600 - x_4 + 200 = 800 - x_4 \leq 800$. The maximum is therefore obtained for $x_4 = 0$ and is 800.

8. (a) (10 points) Find the inverse of the following matrix A and write your answer in the space provided. You must show your work and explain what you are doing to get full marks. (Hint: The inverse will be a matrix with integer entries.)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \qquad A^{-1} =$$

(b) (2 points) Check your answer carefully by multiplying AA^{-1}.

Answer:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \overset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \overset{R_3 \to R_3 + 2R_1}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \overset{R_3 \to -R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \overset{\substack{R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - 3R_3}}{\sim}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \overset{R_1 \to R_1 - 2R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

9. (8 points) (Leontief input-output model) Two companies (A) and (B) use each others services for their production. For each \$1 of output, (A) uses $\$\frac{1}{4}$ of its own products and $\$\frac{1}{2}$ worth of products of (B), while (B) uses for each \$1 of output $\$\frac{2}{5}$ worth of its own products and $\$\frac{1}{2}$ worth of products of (A).

(a) (2 points) Give the consumption matrix of this economy.

Answer: The consumption matrix, with columns and rows ordered A, B is

$$C = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 2/5 \end{bmatrix} = \begin{bmatrix} .25 & .50 \\ .50 & .40 \end{bmatrix}$$

(b) (6 points) What is the total production output $\mathbf{x}$ needed to meet a final demand of $\mathbf{d} = \begin{bmatrix} \$1,000 \\ \$2,000 \end{bmatrix}$? What is the intermediate demand in this case?

Answer: We need to solve $(I - C)\mathbf{x} = \mathbf{d}$, or $\mathbf{x} = (I - C)^{-1}\mathbf{d}$. Using the formula for the inverse of a 2×2 matrix, we have

$$(I - C)^{-1} = \begin{bmatrix} 3/4 & -1/2 \\ -1/2 & 3/5 \end{bmatrix}^{-1} = (9/20 - 1/4)^{-1} \begin{bmatrix} 3/5 & 1/2 \\ 1/2 & 3/4 \end{bmatrix} = 5 \begin{bmatrix} 3/5 & 1/2 \\ 1/2 & 3/4 \end{bmatrix} = \begin{bmatrix} 3 & 5/2 \\ 5/2 & 15/4 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 3 & 5/2 \\ 5/2 & 15/4 \end{bmatrix} \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 3,000 + 5,000 \\ 2,500 + 7,500 \end{bmatrix} = \begin{bmatrix} 8,000 \\ 10,000 \end{bmatrix}$$

is the total output necessary to meet this final demand. The corresponding intermediate demand is

$$C\mathbf{x} = \mathbf{x} - \mathbf{d} = \begin{bmatrix} 8,000 \\ 10,000 \end{bmatrix} - \begin{bmatrix} 1,000 \\ 2,000 \end{bmatrix} = \begin{bmatrix} 7,000 \\ 8,000 \end{bmatrix}.$$

10. (10 points) The matrix $A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & 8 \\ -1 & -2 & 6 \end{bmatrix}$ has exactly two eigenvalues, 1 and 2.

(a) (4 points) Find a basis for the eigenspace corresponding to the eigenvalue 1.

Answer: We need to find a basis of the null space $\text{Nul}\,(A - I_3)$, I_3 the 3×3-identity matrix, which we do by row reducing $A - I_3$:

$$A - I_3 = \begin{bmatrix} 0 & -2 & 4 \\ -2 & -3 & 8 \\ -1 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ -2 & -3 & 8 \\ 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the general solution of the corresponding homogeneous linear system is $x_1 = x_3$, $x_2 = 2x_3$ and hence a basis of the eigenspace $E_1 = \text{Nul}\,(A - I_3)$ is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(b) (4 points) Find a basis for the eigenspace corresponding to the eigenvalue 2.

Answer: We need to find a basis of the null space $\text{Nul}\,(A - 2I_3)$, which we do by row reducing $A - 2I_3$:

$$A - 2I_3 = \begin{bmatrix} -1 & -2 & 4 \\ -2 & -4 & 8 \\ -1 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the general solution of the corresponding homogeneous linear system is $x_1 = -2x_2 + 4x_3$, x_2 and x_3 are free variables. The eigenspace $E_2 = \text{Nul}\,(A - 2I_3)$ has therefore dimension 2, a basis is for example

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

(c) (2 points) Is A diagonalizable? If so, give a diagonalization of A, that is, give matrices P and D such that P is an invertible matrix, D is a diagonal matrix and $A = PDP^{-1}$; if not, explain why not, referring to any theorems discussed in class.

Answer: Yes, A is diagonalizable, because there are 3 linearly independent eigenvectors. For example, we can take P and D as follows:

$$P = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

11. (a) (3 points) Give the definition of p vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ to be linearly independent.

Answer: By definition, the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly independent if for any linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = 0$ all the weights $c_1, c_2, \dots, c_p$ are 0.

(b) (5 points) Let k be a real number. Determine for which value or values of k the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} k \\ 3 \\ 0 \end{bmatrix}$$

are linearly independent.

Answer: Let A be the matrix whose columns are the given vectors. The vectors are linearly independent if and only if the homogeneous linear system with coefficient matrix A has only the trivial solution. We row reduce A and find:

$$A = \begin{bmatrix} 1 & 2 & k \\ 0 & 1 & 3 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & k \\ 0 & 1 & 3 \\ 0 & -2 & -2k \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & k \\ 0 & 1 & 3 \\ 0 & 0 & 6-2k \end{bmatrix}.$$

Thus, the three vectors are linearly independent if only if $6 - 2k \neq 0$, i.e., for $k \neq 3$.

12. (12 points) Suppose the matrix

$$M = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

describes the dynamics of some system, e.g., M is the migration matrix describing the flow of customers between two companies.

(a) (2 points) Check that $\lambda = 1$ and $\lambda = 0.92$ are eigenvalues for that system.

(b) (4 points) Find the corresponding eigenvectors.

(c) (4 points) Represent $\mathbf{x} = \begin{bmatrix} 1/10 \\ 3/2 \end{bmatrix}$ as a linear combination of eigenvectors.

(d) (2 points) Find $M^2\mathbf{x}$.

Answer: (a) We calculate $A - I = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix}$ and $A - 0.92I = \begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix}$. The determinants of these matrices are 0. So, $\lambda = 1$ and $\lambda = 0.92$ are eigenvalues. (b) In order to find the eigenvectors we solve $(A - I)\mathbf{x} = 0$ and $(A - 0.92I)\mathbf{x} = 0$. In the first case the coefficient matrix is

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \sim \begin{bmatrix} -0.05 & 0.03 \\ 0 & 0 \end{bmatrix}$$

so x_2 is free and $x_1 = 3/5x_2$. Hence the eigenvectors for $\lambda = 1$ are the vectors $\begin{bmatrix} \frac{3}{5}x_2 \\ x_2 \end{bmatrix}$ with $0 \neq x_2 \in \mathbb{R}$. For example, the vector $\mathbf{v}_1 = \begin{bmatrix} 3/5 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$. Similarly, for the second case

$$\begin{bmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{bmatrix} \sim \begin{bmatrix} 0.03 & 0.03 \\ 0 & 0 \end{bmatrix}$$

so x_2 is free and $x_1 = -x_2$. Hence the eigenvectors for $\lambda = 0.92$ are the vectors $\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$ with $0 \neq x_2 \in \mathbb{R}$. For example, the vector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 0.92$. (c) In order to find the coordinates c_1, c_2 of $\mathbf{x}_0$ in the basis $\mathbf{v}_1, \mathbf{v}_2$ we solve

$$\begin{bmatrix} 3/5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/2 \end{bmatrix}$$

and get $c_1 = 1$, $c_2 = 1/2 = 0.5$. (d) Now

$$M^2\mathbf{x} = M^2(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = M^2(\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2) = M^2\mathbf{v}_1 + \frac{1}{2}M^2\mathbf{v}_2 == \mathbf{v}_1 + \frac{1}{2}(0.92)^2\mathbf{v}_2$$