MAT 1302 A Mathematical Methods II

Professor: Monica Nevins

Final Exam April 23, 2003

Name (Last, First): ________________________ Student Number: ____________

- This is a closed book exam. Simple calculators (such as Texas Instruments TI-30X series) are permitted; calculators with matrix or graphing capabilities are not permitted.
- This final is out of 71 points; it will count for at least 50% of your final grade.
- You have 3 hours to complete it. There are 11 questions, most with multiple parts.
- Show your work for partial credit where applicable.

Do not write in this space, please:

Question #	Points	Your score
1	8	
2	6	
3	6	
4	4	
5	6	
6	10	
7	4	
8	6	
9	4	
10	7	
11	10	
Total	71	

Q.1. [8 points]

(a) [1 pt] Give an example of a matrix in RREF (reduced row echelon form).

(b) [1 pt] Compute $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

(c) [3 pts] Give 3 elements in span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) [1 pt] Given $z = 3 + 4i$ compute $|z|$.

(e) [2 pts] Give the roots of the polynomial $x^2 - 4x + 8$.

Q.2. [6 points]

This question has no partial credit.
The following is the augmented matrix of a system of linear equations, where k a parameter:

$$\begin{bmatrix} 1 & 2 & 4 & | & 2 \\ 0 & k+1 & 4 & | & 2 \\ 0 & 0 & k-1 & | & -1 \end{bmatrix}$$

(a) [**2 pts**] For which of the following values of k is the system inconsistent:

$-1, \qquad 0, \qquad 1, \qquad$ no values

Answer: ____________

(b) [**2 pts**] For which of the following values of k does the system have infinitely many solutions:

$-1, \qquad 0, \qquad 1, \qquad$ no values

Answer: ____________

(c) [**2 pts**] For which of the following values of k does the system have a unique solution:

$-1, \qquad 0, \qquad 1, \qquad$ no values

Answer: ____________

Q.3. [6 points]

(a) [1 pt] Define $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

(b) [5 pts] For which value(s) of the parameter k is the vector $\begin{bmatrix} 1 \\ k \\ k \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$?

Q.4. [4 points]

Are the following vectors linearly independent or linearly dependent?

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

Q.5. [6 points]

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \qquad A^{-1} = \underline{\hspace{4cm}}$$

Q.6. [10 points]

Note: each answer is worth 1 point, up to a maximum of 10 points.

(a) Suppose A is an invertible 6×6 matrix. Answer the following questions about A.

1 What is the dimension of the column space of A? ____________

2 What is the dimension of the null space of A? ____________

3 Are the columns of A necessarily linearly independent? ____________

4 Is it necessarily true that the columns of A span $\mathbb{R}^n$? ____________

5 Is $\det(A) = 0$? ____________

6 What is the rank of A? ____________

7 Is A necessarily diagonalizable? ____________

(b) Suppose B is a 5×7 matrix such that $rank(B) = 5$. Answer the following questions about B.

1 Is B is invertible? ____________

2 What is $\dim(Nul(B))$? ____________

3 How many pivot columns are there in the RREF of B? ____________

4 Do the columns of B form a basis for $Col(B)$? ____________

5 Will there be any zero rows in the RREF of B? ____________

Q.7. [4 points]
Given the following consumption matrix for an open economy with two sectors:

$$C = \begin{bmatrix} .5 & .3 \\ .1 & .2 \end{bmatrix}.$$

(a) [2 pts] Give the equation relating total production $\vec{x}$ and the demand $\vec{d}$ of the open sector (= surplus production) for this economy.

(b) [2 pts] Given the total production is $\vec{x} = \left[\begin{smallmatrix} 200 \\ 100 \end{smallmatrix}\right]$, what is the surplus (or final demand) $\vec{d}$? (No partial credit.)

Answer: ____________

Q.8. [6 points]
Given the following matrix:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 5 \\ 3 & 1 & 5 \\ -1 & 4 & -6 \end{bmatrix}$$

whose reduced row echelon form (RREF) is:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

answer the following questions. (No partial credit.)

(a) [2 pts] Give a basis for $Col(A)$.

(b) [2 pts] Give a basis for $Nul(A)$.

(c) [1 pt] $\dim Col(A) =$ ____________

(d) [1 pt] $\dim Nul(A) =$ ____________

Q.9. [4 points]

Suppose that $\vec{u}$ and $\vec{v}$ are two eigenvectors of a matrix A corresponding to the same eigenvalue λ. Prove that $\vec{u} + \vec{v}$ is also an eigenvector of A corresponding to the eigenvalue λ.

Q.10. [7 points]

(a) [2 pts] Give the **characteristic polynomial** of the matrix

$$A = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}.$$

(b) [5 pts] The matrix $A = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$ has complex eigenvalues $\lambda = 3 \pm i$. Find a basis for each eigenspace of A.

Q.11. [10 points]

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & 1 \\ 3 & 3 & -3 \end{bmatrix}$. It has exactly two eigenvalues, -2 and -3.

(a) [**4 pts**] Find a basis for the eigenspace corresponding to the eigenvalue -2.
(b) [**4 pts**] Find a basis for the eigenspace corresponding to the eigenvalue -3.
(c) [**2 pts**] Is A diagonalizable? If so, give a diagonalization of A (that is, give matrices P and D such that $A = PDP^{-1}$); if not, explain why not, referring to any theorems discussed in class.

You may use the next page also if necessary.

Q. 11, ctd. Recall that $A = \begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & 1 \\ 3 & 3 & -3 \end{bmatrix}$ and the eigenvalues are -2 and -3.

This page intentionally blank.