MAT 1302 A Mathematical Methods II

Final Exam: Solutions and Marking Scheme April 23, 2003

Q.1. [8 points]

(a) [1 pt] Give an example of a matrix in RREF (reduced row echelon form).

Marking:1 point for any correct answer (1's as pivots, echelon, zeros above, below and to the left of any pivot.)

(b) [1 pt] Compute $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Answer: $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

Marking:1 point correct answer

(c) [3 pts] Give 3 elements in span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Marking:1 point each correct vector; if you gave more than 3, only the first three are checked.

(d) [1 pt] Given $z = 3 + 4i$ compute $|z|$.

Answer: $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Marking:1 point correct answer, deduct .5 for answer ± 5.

(e) [2 pts] Give the roots of the polynomial $x^2 - 4x + 8$.

Answer: The quadratic formula gives

$$x = \frac{4 \pm \sqrt{16 - 4(8)}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

as the two roots.

Marking:1 point for correct quadratic formula, 1 point for correct simplification; or 2 point for correct answer by any means.

Q.2. [6 points]

This question has no partial credit.

The following is the augmented matrix of a system of linear equations, where k a parameter:

$$\begin{bmatrix} 1 & 2 & 4 & | & 2 \\ 0 & k+1 & 4 & | & 2 \\ 0 & 0 & k-1 & | & -1 \end{bmatrix}$$

(a) [**2 pts**] For which of the following values of k is the system inconsistent:

$$-1, \qquad 0, \qquad 1, \qquad \text{no values}$$

Answer: $k = 1$ (because last row is 0=1)

(b) [**2 pts**] For which of the following values of k does the system have infinitely many solutions:

$$-1, \qquad 0, \qquad 1, \qquad \text{no values}$$

Answer: $k = -1$ (because then 2nd and 3rd rows are multiples, the system is consistent, and there is a free variable)

(c) [**2 pts**] For which of the following values of k does the system have a unique solution:

$$-1, \qquad 0, \qquad 1, \qquad \text{no values}$$

Answer: $k = 0$ (lots of other values are ok, too; it gives 3 pivots in the coefficient matrix)

Marking: No partial credit; 2 points each for correct answer. (Circling more than one answer is incorrect = 0 points.)

Q.3. [**6 points**]

(a) [**1 pt**] Define $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

Answer:$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_1, c_2 \text{ are scalars}\}$

Marking:1 point for correct definition; .5 for something mostly correct.

(b) [**5 pts**] For which value(s) of the parameter k is the vector $\begin{bmatrix} 1 \\ k \\ k \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$?

Answer: We have to row reduce the following augmented matrix and determine the values of k for which the system is consistent:

$$\begin{bmatrix} 1 & 3 & | & 1 \\ 2 & 2 & | & k \\ 3 & 1 & | & k \end{bmatrix} \underset{\substack{-2R1+R2 \\ -3R1+R3}}{\simeq} \begin{bmatrix} 1 & 3 & | & 1 \\ 0 & -4 & | & k-2 \\ 0 & -8 & | & k-3 \end{bmatrix} \underset{-2R2+R3}{\simeq} \begin{bmatrix} 1 & 3 & | & 1 \\ 0 & -4 & | & k-2 \\ 0 & 0 & | & -2(k-2)+k-3 \end{bmatrix}$$

This is in echelon form. The system is consistent if and only if $0 = -2(k-2)+k-3 = -k+1$, in other words, if and only if $k = 1$.

Marking:1 point for set up of augmented matrix; 2 points for the row reduction; 1 point for the correct interpretation of what you need; 1 point for correct final answer.

Q.4. [4 points]

Are the following vectors linearly independent or linearly dependent?

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

Answer: We need to determine the nature of the solutions to the dependence equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$; this amounts to reducing the following matrix:

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} R1 \leftrightarrow R2 \\ \simeq \\ \ \end{matrix} \begin{bmatrix} 1 & -1 & 5 \\ 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} \simeq \\ -3R1 + R2 \\ -2R1 + R3 \end{matrix} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & -15 \\ 0 & 3 & -9 \end{bmatrix}$$

$$\begin{matrix} \simeq \\ \frac{1}{5}R2 \\ -3new(R2) + R3 \end{matrix} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence there is no pivot in the last column, so there are infinitely many solutions, so the vectors are linearly dependent.

Marking:1 point for the matrix to reduce, 2 points for the row reduction, 1 point for the final answer (based on your final matrix)

Q.5. [6 points]

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \qquad A^{-1} = \underline{\qquad\qquad}$$

Answer: We have to reduce $[A|I]$ all the way to $[I|A^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 2 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R2 + R1 \\ \simeq \\ \ \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \simeq \\ R1 + R2 \\ -2R1 + R3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 1 & 0 \\ 0 & 3 & 6 & 1 & 2 & 0 \\ 0 & -3 & -7 & -2 & -2 & 1 \end{array}\right]$$

$$\begin{matrix} -R2 + R1 \\ \frac{1}{3}R2 \\ R2 + R3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} -2R3 + R1 \\ 2R3 + R2 \\ -R3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -2 \\ 0 & 1 & 0 & \frac{-5}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -1 & -2 \\ \frac{-5}{3} & \frac{2}{3} & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

We check quickly that this is correct by multiplying out $AA^{-1} = I$.

Marking:1 point for the setup $[A|I]$; 4 points for the row reduction (deducting 1 point for each error; but no partial credit towards an incorrect answer if row operations aren't indicated); 1 point for giving an answer which is a 3×3 matrix.

Q.6. [10 points]

Note: each answer is worth 1 point, up to a maximum of 10 points.
(a) Suppose A is an invertible 6×6 matrix. Answer the following questions about A.

1 What is the dimension of the column space of A? <u>6</u>

2 What is the dimension of the null space of A? <u>0</u>

3 Are the columns of A necessarily linearly independent? <u>yes</u>

4 Is it necessarily true that the columns of A span $\mathbb{R}^6$? <u>yes</u>

5 Is $\det(A) = 0$? <u>no</u>

6 What is the rank of A? <u>6</u>

7 Is A necessarily diagonalizable? <u>no</u>

(b) Suppose B is a 5×7 matrix such that $rank(B) = 5$. Answer the following questions about B.

1 Is B is invertible? <u>no (not square!)</u>

2 What is $\dim(Nul(B))$? <u>$7 - 5 = 2$</u>

3 How many pivot columns are there in the RREF of B? <u>5</u>

4 Do the columns of B form a basis for $Col(B)$? <u>no (too many)</u>

5 Will there be any zero rows in the RREF of B? <u>no</u>

Marking: As indicated, each answer is 1 point for correct, 0 points for incorrect, up to a maximum of 10. Thus, you got 2 free mistakes.

Q.7. [4 points]
Given the following consumption matrix for an open economy with two sectors:

$$C = \begin{bmatrix} .5 & .3 \\ .1 & .2 \end{bmatrix}.$$

(a) [2 pts] Give the equation relating total production $\vec{x}$ and the demand $\vec{d}$ of the open sector (= surplus production) for this economy.

Answer: $\vec{x} = C\vec{x} + \vec{d}$, or $(I - C)\vec{x} = \vec{d}$.

Marking: Some version of the above (perhaps with the actual matrix C from above) is worth 2 points. Something like $C\vec{x} = \vec{d}$, or any other algebraically correct equation, is 1 point. (So $C = \vec{x} + \vec{d}$ is 0.)
(b) [2 pts] Given the total production is $\vec{x} = \left[\begin{smallmatrix} 200 \\ 100 \end{smallmatrix}\right]$, what is the surplus (or final demand) $\vec{d}$? (No partial credit.)

Answer: $I - C = \begin{bmatrix} .5 & -.3 \\ -.1 & .8 \end{bmatrix}$ so $\vec{d} = (I - C)\vec{x} = \begin{bmatrix} 70 \\ 60 \end{bmatrix}$.

Marking: 2 points for correct answer.

Q.8. [6 points]
Given the following matrix:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 5 \\ 3 & 1 & 5 \\ -1 & 4 & -6 \end{bmatrix}$$

whose reduced row echelon form (RREF) is:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

answer the following questions. (No partial credit.)

(a) [2 pts] Give a basis for $Col(A)$.

Answer: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 4 \end{bmatrix} \right\}$

(b) [2 pts] Give a basis for $Nul(A)$.

Answer: Augmenting the RREF by the invisible zero column, we have $x_1 = -2x_3$, $x_2 = x_3$ and x_3 is free. Thus $(x_1, x_2, x_3) = x_3(-2, 1, 1)$ in parametric vector form, so a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) [1 pt] $\dim Col(A) = \underline{2}$

(d) [1 pt] $\dim Nul(A) = \underline{1}$

Marking: No partial credit towards incorrect answers.

Q.9. [4 points]

Suppose that $\vec{u}$ and $\vec{v}$ are two eigenvectors of a matrix A corresponding to the same eigenvalue λ. Prove that $\vec{u} + \vec{v}$ is also an eigenvector of A corresponding to the eigenvalue λ.

Answer: We are given that $A\vec{u} = \lambda\vec{u}$ and $A\vec{v} = \lambda\vec{v}$. We need to show that $A(\vec{u}+\vec{v}) = \lambda(\vec{u} + \vec{v})$.

So evaluate: $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \lambda\vec{u} + \lambda\vec{v} = \lambda(\vec{u} + \vec{v})$, as required.

Marking: 2 points towards correct interpretation of hypotheses, 2 points towards reaching the conclusion.

Q.10. [7 points]

(a) [2 pts] Give the **characteristic polynomial** of the matrix

$$A = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}.$$

Answer: $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{bmatrix} = (1 - \lambda)(3 - \lambda) + 10 = \lambda^2 - 4\lambda + 13.$

Marking:1 point for correctly considering $\det(A-\lambda I)$ (and not $\det(A)$, for example); 1 point for correctly evaluating the determinant. (Thus it didn't matter if you incorrectly subtracted $A-\lambda I$, or incorrectly simplified the polynomial.)

(b) [5 pts] The matrix $A = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$ has complex eigenvalues $\lambda = 3 \pm i$. Find a basis for each eigenspace of A.

Answer: We can start with either eigenvalue; let's take $\lambda = 3+i$.

$$A - \lambda I = \begin{bmatrix} 2-(3+i) & -2 \\ 1 & 4-(3+i) \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \simeq \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

where this last step was either by explictly row reducing, or by realizing that the RREF must have a zero row. Thus $Nul(A-\lambda I) = \{(-(1-i)x_2, x_2) | x_2 \text{ is free}\}$, so a basis for the $(3+i)$-eigenspace is $\vec{x} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$. Thus a basis for the $(3-i)$-eigenspace is the complex conjugate of this vector, namely $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$.

Marking:1 point for writing $A-\lambda I$, for one choice of λ; 1 point for its RREF (or some simplified form from which the answer is deduced); 1 point for relating x_1 and x_2; 1 point for a basis for the eigenspace; 1 point for the complex conjugate case.

Q.11. [10 points]

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & 1 \\ 3 & 3 & -3 \end{bmatrix}$. It has exactly two eigenvalues, -2 and -3.

(a) [4 pts] Find a basis for the eigenspace corresponding to the eigenvalue -2.
(b) [4 pts] Find a basis for the eigenspace corresponding to the eigenvalue -3.
(c) [2 pts] Is A diagonalizable? If so, give a diagonalization of A (that is, give matrices P and D such that $A = PDP^{-1}$); if not, explain why not, referring to any theorems discussed in class.

Answer:(a) We need to find a basis for $Nul(A-\lambda I)$.

$$A - (-2)I = A + 2I = \begin{bmatrix} 3 & 3 & -1 \\ -3 & -3 & 1 \\ 3 & 3 & -1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -x_2 + \frac{1}{3}x_3$ and x_2, x_3 are free. Hence a basis for $Nul(A+2I)$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Now the same for $Nul(A+3I)$:

$$A-(-3)I = A+3I = \begin{bmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 3 & 3 & 0 \end{bmatrix} \underset{\simeq}{R2+R1} \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 3 & 3 & 0 \end{bmatrix} \begin{matrix} \simeq \\ 3R1+R2 \\ -3R1+R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\simeq}{-R2+R1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $x_1 = x_3$, $x_2 = -x_3$ and x_3 is free. Thus a basis for the null space is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(c) Yes, it's diagonalizable, because there are 3 linearly independent eigenvectors. For example, we can take

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}.$$

Marking: For parts (a) and (b), it's 1 point for $A - \lambda I$ written out; 1 point for the correct RREF; 1 point for interpreting the solution as vectors in $\mathbb{R}^3$, and 1 point for a basis. For part (c), there are many correct answers (since you can order the eigenvalues differently, and scale the basis vectors to integers, or not), any of which is worth 2 points for totally correct, 1 point for mostly correct. If you got only 1 vector in the basis for part (a), then you could still get 2 points here if you explained lucidly why the matrix was not diagonalizable based on your evidence.