Mat 1302.W99 Final Solutions

1. Find all k for which the following system has infinitely many solutions.

$$\begin{aligned} x + 2y + z &= 0 \\ x + 3y + 2kz &= 0 \\ 2x + 3y + kz &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2k \\ 2 & 3 & k \end{bmatrix}$$

A. 0
B. 1/2
C. 3
D. -3
E. 1
F. -1

$$\det A = \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2k-1 \\ 0 & -1 & k-2 \end{bmatrix} = \begin{vmatrix} 1 & 2k-1 \\ -1 & k-2 \end{vmatrix}$$

$$= (k-2) + (2k-1)$$

$$= 3k - 3$$

$$= 0$$

$$\Leftrightarrow k = 1.$$

2. The coefficient matrix A in a homogeneous system of 12 equations in 16 unknowns is known to have rank 6. How many parameters are there in the general solution?

A. 16
B. none
C. 10
D. 4
E. 12
F. 6

16

12

rank A = 6

$\therefore$ # leading ones = 6

$\therefore$ # parameters = # vars − # leading ones

= 16 − 6

= 10

3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$. What is the second row of A^{-1} ?

A. [13/8, -1/2, -1/8]
B. [-15/8, 1/2, 3/8]
C. [-15/8, 1/8, 3/8]
D. [-15, 4, 3]
E. [-1/2, 1/2, 0]
F. [-15/4, 1, 3/4]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} \times & \times & \times & \times & \times & \times \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

4. Let A and B be 4×4 matrices with det A = 3 and det B = -5. Find $\det(2AB^tB^{-2})$.

A. 48/3
B. -36/5
C. 24/5
D. -48/5
E. 12/5
F. -24/5

$$\det(2AB^tB^{-2}) = 2^4 \cdot \det A \cdot \det B \cdot \frac{1}{(\det B)^2}$$

$$= 16 \cdot 3 \cdot \frac{1}{-5}$$

$$= -\frac{48}{5}$$

5. Let A be an $n \times n$ matrix. Among the following statements, one is not equivalent to the other four. Which one is it?

- A. The ~~column~~ rank of A is n.
- B. A is not invertible.
- C. The equation $Ax = b$ has a unique solution x for any n-vector b.
- D. A can be row-reduced to the identity matrix I_n.
- E. The rows of A are linearly independent.

B.

6. Which two of the following are <u>not</u> subspaces of R^4?

$R = \{(a, b, c, d) \mid c = a + 2b, d = a - 3b\}$
$S = \{(a, b, c, d) \mid a = 0, b = 0\}$
} sets of solutions of homog. linear systems & ∴ subspaces.
$T = \{(a, b, c, d) \mid a - b = 2, c = d\}$
$U = \{(a, b, c, d) \mid a \geq 0, b \geq 0\}$

- A. T and U
- B. S and U
- C. R and T
- D. R and U
- E. S and T
- F. T and U

7. What is the dimension of the subspace spanned by S = { (1, 1, 1), (-1, 1, -1), (1, 1, 3), (0, 2, 1) } ?

A. 0
B. 3
C. 1
D. 2
E. 4
F. These vectors do not span a subspace

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ } 3 vectors

R.E. form

$\therefore$ dim = 3

8. A basis for the eigenspace corresponding to the eigenvalue 1 of the matrix $\begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = A$ is:

A. { (-1, 4, 1) }
B. { (-1, 4, 1), (1, -1, -1) }
C. { (1, 2, 1) }
D. { (-2, 1, 3) }
E. { (1, 2, 1), (-1, 4, 1) }
F. { (1, -1, -1) }

$$[A-I\,|\,0] = \left[\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \quad -R_3+R_2 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & +1 & 0 \\ 0 & -1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \overset{r}{1} & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = -r$
$y = 4r$
$z = r$

$\therefore \{(-1, 4, 1)\}$ is a basis of the eigenspace.

9. Consider the data points $(1,8), (3,26), (5,60)$

a) Find a polynomial of degree two whose graph passes through the given points.
b) How many polynomials of degree 3 will pass through the given points? You do not need to find them to answer, but you must explain your answer.
c) Using your result from (a), predict the value of y when $x = 2$.

3 a) Let $p(x) = a_0 + a_1 x + a_2 x^2$

$(1,8)$ on the graph $\Leftrightarrow$
$(3,26)$ " $\Leftrightarrow$
$(5,60)$ " $\Leftrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 26 \\ 1 & 5 & 25 & 60 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 2 & 8 & 18 \\ 0 & 4 & 24 & 52 \end{array}\right]$$

+ 1 for 1 correct row opⁿ

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 8 & 16 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{c|c} I_3 & \begin{array}{c} 5 \\ 1 \\ 2 \end{array} \end{array}\right].$$

Hence + 1 correct answer
$p(x) = 5 + x + 2x^2$

1 b) $p(2) = \underline{\underline{15}}$

2 c) If $q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ passes through those pts, we obtain the system

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 26 \\ 1 & 5 & 25 & 125 & 60 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & * & 5 \\ 0 & 1 & 0 & * & 1 \\ 0 & 0 & 1 & * & 2 \end{array}\right]$$

So a_3 can be a parameter; hence there are ∞ly many possibilities.

10. Consider an economy consisting of two industries with input-output matrix given below.

$$A = \begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

a) What is the amount of production available to the open sector if the total output levels in the industries are given by $X = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$?

b) What is the production level necessary to meet the open sector demand given by $D = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$?

c) What is the meaning of the (2,1) entry of the matrix above?

2 a) $\underbrace{D = X - AX}_{1} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} .1 & .4 \\ .3 & .2 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 5 \end{bmatrix}}_{1}$

3 b) $\underbrace{X = (I-A)^{-1} D}_{1}$; $I - A = \begin{bmatrix} .9 & -.4 \\ -.3 & .8 \end{bmatrix}$, so $(I-A)^{-1} = \frac{1}{.6}\begin{bmatrix} .8 & .4 \\ .3 & .9 \end{bmatrix}$

$= \begin{bmatrix} 4/3 & 2/3 \\ 1/2 & 3/2 \end{bmatrix}$ (1) $\therefore$ $X = \begin{bmatrix} 4/3 & 2/3 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 28 \\ 18 \end{bmatrix}$. (1)

1 c) It is the amount (in \$) of the output of industry 2 required to produce \$1 of the output of industry 1.

11. The following table gives occupational transistion probabilities.

Initial generation			
white-collar	manual		
1	.2	white-collar	**Next generation**
0	.8	manual	

a) Give the matrix of transition probabilities.

b) If the mother is a manual worker, what is the probability that her daughter or son will be a white-collar worker?

c) If there are 10,000 in the white-collar category and 20,000 in the manual category, what will the distribution be one generation later?

d) If the father is a manual worker, what is the probability that his grandaughter will be a manual worker?

e) What is the long-term distribution likely to be?

1 a) $P = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$

1 b) $.2$

1 c) $P\begin{bmatrix} 10000 \\ 20000 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 10000 \\ 20000 \end{bmatrix} = \begin{bmatrix} 14000 \\ 16000 \end{bmatrix}$

1 d) 2 generations later: $P^2 = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}\begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} = \begin{bmatrix} 1 & .36 \\ 0 & .64 \end{bmatrix}$

$\therefore$ $\underline{\underline{.64}}$ or 64%

2 e) We solve $\underbrace{PX = X}_{1}$: $[P-I|0] = \left[\begin{array}{cc|c} 0 & .2 & 0 \\ 0 & -.2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \therefore \begin{array}{l} x = s \\ y = 0 \end{array}$

and $x+y = 30000$ $\therefore$ 30000 white collar workers
0 manual workers $\begin{bmatrix} 30000 \\ 0 \end{bmatrix}$. 1

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$.

a) Find the characteristic polynomial of A.

b) Show that the eigenvalues of A are 1 and 3.

c) Find a basis for the eigenspace E_1 corresponding to $\lambda = 1$. What is its dimension? Give a geometric description of E_1.

d) Find a basis for the eigenspace E_3 corresponding to $\lambda = 3$. What is its dimension? Give a geometric description of E_3.

1 a) $C_A(x) = |A - xI| = \begin{vmatrix} 1-x & 0 & 0 \\ -2 & 1-x & 0 \\ -2 & 0 & 3-x \end{vmatrix} \overset{\text{row 1}}{=} (1-x)\begin{vmatrix} 1-x & 0 \\ 0 & 3-x \end{vmatrix} = (1-x)^2(3-x)$

1 b) $C_A(x) = 0 \iff x = 1$ or $x = 3$.

2 c) $E_1 = \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{v} \}$: $[A - I \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \sim \begin{array}{cccc} & s & t & \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

$\therefore$ $x = t$, $y = s$, $z = t$ so $\{(1,0,1), (0,1,0)\}$ is a basis for E_1. 1

$\therefore$ $\dim E_1 = 2$ ½, and so E_1 is a plane in $\mathbb{R}^3$ through O. ½

2 d) E_3 : $[A - 3I \mid 0] = \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim\sim \begin{array}{c} \quad\quad s \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\therefore$ $x = 0$, $y = s$, $z = s$

$\therefore$ $\{(0,1,1)\}$ 1 is a basis for E_3 $\therefore$ $\dim E_3 = 1$ ½,

so E_3 is a line ½ in $\mathbb{R}^3$ through O.

13. Consider the homogeneous system of equations

$$\begin{array}{rcrcrcrcl} 2x_1 & - & 4x_2+ & + & 12x_3 & - & 10x_4 & = & 0 \\ -1x_1 & + & 2x_2 & - & 3x_3 & + & 2x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 9x_3 & - & 6x_4 & = & 0 \end{array}$$

a) Write down the augmented matrix and perform the first TWO steps in the Gaussian algorithm. DO NOT COMPLETE THE ROW REDUCTION.

b) The Gaussian algorithm results in the following augmented matrix. Write down the general solution to the system of equations.

$$\begin{array}{c} \qquad\; s \qquad\qquad\qquad \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

c) Let U be the set of all solutions to the above system of equations. Explain briefly why U is a subspace of $\mathbf{R}^4$.

d) Let A be the coefficient matrix of the above system of equations.
   * What is the rank of A?
   * Does $Ax = b$ have a solution for every $b \in \mathbf{R}^3$?
   * Explain how these two questions are related.

a) $$\left[\begin{array}{cccc|c} 2 & -4 & 12 & -10 & 0 \\ -1 & 2 & -3 & 2 & 0 \\ 2 & -4 & 9 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} -1 & 2 & -3 & 2 & 0 \\ 2 & -4 & 12 & -10 & 0 \\ 2 & -4 & 9 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 2 & -4 & 12 & -10 & 0 \\ 2 & -4 & 9 & -6 & 0 \end{array}\right]$$

$$\left(\sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 0 \\ 0 & 0 & 6 & -6 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right] \sim \dots \right)$$

b) $$\begin{aligned} x_1 &= 2s \\ x_2 &= s \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned} \quad , s \in \mathbb{R}$$

c) The system is a homogeneous system, so its set of solutions is a subspace of $\mathbb{R}^4$. (or, directly: the set of solutions, by (b), is $\{s(2,1,0,0) \mid s \in \mathbb{R}\} = \text{span}\{(2,1,0,0)\}$, again a subspace, since it's a span.)

d). By the matrix in b), rank $A = 3$

• Yes, because $[A|b] \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$ which is always consistent.

(or : rank$[A|b] \leq 3$, and rank$A = 3 \Rightarrow$ rank A = rank$[A|b]$.)

• see ↑ or; rank $A = 3 \Rightarrow$ there's a leading 1 in each row, so there's no opportunity for an inconsistent system.

14. Consider the vectors $v_1 = (1, -2, -4)$, $v_2 = (2, -3, 1)$, and $v_3 = (3, -5, -3)$. Justify your answer to each of the following questions.

a) Does $\{v_1, v_2, v_3\}$ span $\mathbf{R}^3$?

b) Is $\{v_1, v_2, v_3\}$ linearly independent?

c) Let V be the subspace of $\mathbf{R}^3$ generated by the vectors v_1, v_2 and v_3. What is the dimension of V? Give a geometric description of V.

a) $\begin{vmatrix} 1 & 2 & 3 \\ -2 & -3 & -5 \\ -4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 9 & 9 \end{vmatrix} = 0$, so these 3 vectors do not span $\mathbb{R}^3$.

b) Since 3 vectors in $\mathbb{R}^3$ are l.i. $\Leftrightarrow$ they span $\mathbb{R}^3$, part (a) shows these vectors aren't l.i..

c) $\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & 1 \\ 3 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 9 \\ 0 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \cdots \begin{bmatrix} 1 & 0 & 14 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$ Hence $\{(1,0,14), (0,1,9)\}$ is a basis for V (the row space above)

So $\dim V = 2$. V is a plane through 0.