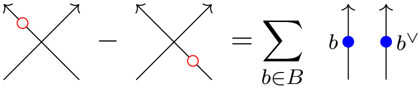
Heisenberg categorification



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Slides available online: alistairsavage.ca/talks

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Outline

Goals:

1. Explain diagrammatic categorification
2. Describe a family of categories that categorify the Heisenberg algebra

Overview:

1. Strict monoidal categories and string diagrams
2. Monoidally presented algebras
3. Adjunction and pivotal categories
4. The Frobenius Heisenberg category

Strict monoidal categories

A strict monoidal category is a category $\mathcal{C}$ equipped with

- a bifunctor (the tensor product) $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
- a unit object $\mathbf{1}$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbf{1} \otimes A = A = A \otimes \mathbf{1}$ for all objects A.

Remark: Non-strict monoidal categories

In a (not necessarily strict) monoidal category, the equalities above are replaced by isomorphism, and we impose some coherence conditions.

Every monoidal category is monoidally equivalent to a strict one.

$\Bbbk$-linear monoidal categories

Fix a commutative ground ring $\Bbbk$.

A strict $\Bbbk$-linear monoidal category is a strict monoidal category such that

- each morphism space is a $\Bbbk$-module,
- composition of morphisms is $\Bbbk$-bilinear,
- tensor product of morphisms is $\Bbbk$-bilinear.

The interchange law

The axioms of a strict monoidal category imply the interchange law: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{tikzcd}
A_1 \otimes B_1 \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, "f \otimes \mathrm{id}"'] \arrow[dr, "f \otimes g"] & A_1 \otimes B_2 \arrow[d, "f \otimes \mathrm{id}"] \\
A_2 \otimes B_1 \arrow[r, "\mathrm{id} \otimes g"'] & A_2 \otimes B_2
\end{tikzcd}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbf{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

Example (Associative algebras)

A (strict) $\Bbbk$-linear monoidal category is simply a commuative associative unital $\Bbbk$-algebra.

Categorification via split Grothendieck group

Suppose $\mathcal{C}$ is an additive category (i.e. have $\oplus$).

$\mathrm{Iso}_{\mathbb{Z}}(\mathcal{C})$ = free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K_0(\mathcal{C}) = \mathrm{Iso}_{\mathbb{Z}}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$

If $\mathcal{C}$ is monoidal, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to categorify a ring R is to find an additive monoidal category $\mathcal{C}$ such that

$$K_0(\mathcal{C}) \cong R \quad \text{as rings.}$$

The Heisenberg algebra

Let $\mathfrak{h}$ be the infinite-dimensional Heisenberg Lie algebra.

Thus, $\mathfrak{h}$ is the complex Lie algebra with basis

$$\{c, q_n^\pm : n \geq 1\}$$

and product

$$[q_m^+, q_n^+] = [q_m^-, q_n^-] = [c, q_n^\pm] = 0, \qquad [q_m^+, q_n^-] = \delta_{m,n} c.$$

The associative Heisenberg algebra at central charge $\xi \in \mathbb{Z}$ is

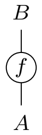
$$U(\mathfrak{h})/\langle c - \xi \rangle.$$

We will describe categories that categorify these algebras.

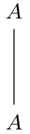
String diagrams

Let's draw pictures! Fix a strict monoidal category $\mathcal{C}$.

We will denote a morphism $f\colon A \to B$ by:



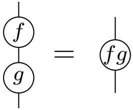
The identity map $\mathrm{id}_A\colon A \to A$ is a string with no label:

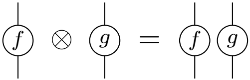


We sometimes omit the object labels when they are clear or unimportant.

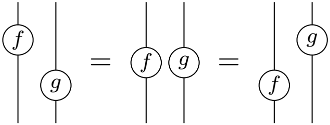
String diagrams

Composition is vertical stacking and tensor product is horizontal juxtaposition:

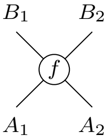




The interchange law then becomes:



A morphism $f\colon A_1 \otimes A_2 \to B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give presentations of some strict $\Bbbk$-linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection A_i, $i \in I$, then we have all possible tensor products of these objects:

$$\mathbf{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection f_j, $j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$\mathrm{id}_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some relations on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

Monoidally generated symmetric groups

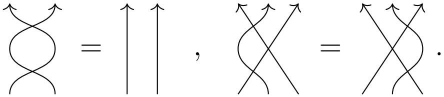
Define a strict monoidal category $\mathcal{S}$ with one generating object Q_+ and denote

$$\mathrm{id}_{\mathsf{Q}_+} = \uparrow$$

We have one generating morphism

 $: \mathsf{Q}_+ \otimes \mathsf{Q}_+ \to \mathsf{Q}_+ \otimes \mathsf{Q}_+.$

We impose the relations:



It is straightforward to verify that

$$\mathrm{End}_{\mathcal{S}}(\mathsf{Q}_+^{\otimes n}) = S_n$$

is the symmetric group on n letters.

Monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

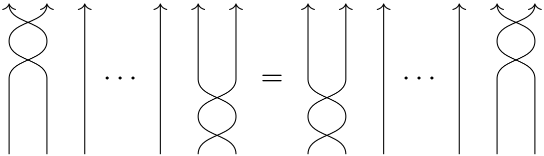
- one generating morphism, and
- two relations,

to get all the symmetric groups.

Note that the "distant braid relation"

$$s_i s_j = s_j s_i, \qquad |i-j|>1$$

for simple transpositions follows for free from the interchange law:



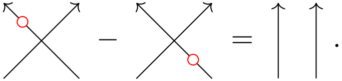
Note: If we define $\mathcal{S}$ to be $\Bbbk$-linear, then $\mathrm{End}_{\mathcal{S}}(\mathsf{Q}_+^{\otimes n}) = \Bbbk S_n$.

The degenerate affine Hecke category

Start again with the strict $\Bbbk$-linear monoidal category $\mathcal{S}$, but add a morphism:

$$: \mathrm{Q}_+ \to \mathrm{Q}_+$$

We impose the additional relation:



Now

$$\mathrm{End}(\mathrm{Q}_+^{\otimes n})$$

is the degenerate affine Hecke algebra (of type A).

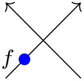
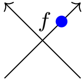
The wreath product category

Fix an associative $\Bbbk$-algebra F. We add an endomorphism of Q_+ for each element of F.

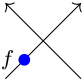
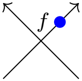
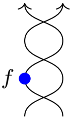
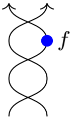
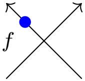
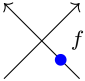
More precisely, let $\mathcal{W}(F)$ be the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{S}$ by adding morphisms such that we have an algebra homomorphism:

$F \to \operatorname{End} \mathsf{Q}_+, \qquad f \mapsto$ 

We impose the additional relations:

 $=$  $, \quad f \in F$

Example of diagrammatic proof:

 $=$  $\implies$  $=$  $\implies$  $=$ 

The wreath product category

$$\mathrm{End}_{\mathcal{W}(F)}(\mathsf{Q}_+^{\otimes n}) = F^{\otimes n} \rtimes S_n$$

is a wreath product algebra.

As a vector space,

$$F^{\otimes n} \rtimes S_n = F^{\otimes n} \otimes_{\Bbbk} \Bbbk S_n.$$

Multiplication is determined by

$$(f_1 \otimes \pi_1)(f_2 \otimes \pi_2) = f_1(\pi_1 \cdot f_2) \otimes \pi_1\pi_2, \quad f_1, f_2 \in F^{\otimes n},\ \pi_1, \pi_2 \in S_n,$$

where $\pi_1 \cdot f_2$ denotes the natural action of S_n on $F^{\otimes n}$ by permutation of the factors.

Note: $\mathcal{W}(\Bbbk) = \mathcal{S}$, the symmetric group category.

Want: An affine version of the wreath product category. $F = \Bbbk$ should recover the degenerate affine Hecke category.

Frobenius algebras: Definition

Frobenius algebra

A Frobenius algebra is a f.d. associative algebra F together with a linear trace map

$$\mathrm{tr}\colon F \to \Bbbk$$

such that the induced map

$$F \to \mathrm{Hom}_{\Bbbk}(F, \Bbbk), \quad f \mapsto (g \mapsto \mathrm{tr}(gf)),$$

is an isomorphism.

For simplicity, we assume that the trace is symmetric:

$$\mathrm{tr}(fg) = \mathrm{tr}(gf), \quad \text{for all } f, g \in F.$$

Frobenius algebras: Examples

Example ($\mathbb{k}$)

$\mathbb{k}$ is a Frobenius algebra with $\mathrm{tr} = \mathrm{id}_{\mathbb{k}}$.

Example (Matrix algebra)

Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example ($\mathbb{k}[x]/(x^k)$)

$\mathbb{k}[x]/(x^k)$ is a Frobenius algebra with

$$\mathrm{tr}(x^\ell) = \delta_{\ell,k-1}.$$

Frobenius algebras: Examples

Example (Group algebra)

Suppose G is a finite group.

The group algebra $\Bbbk G$ is a Frobenius algebra with

$$\mathrm{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$

Example (Zigzag algebra)

Associated to every quiver is a zigzag algebra. These are Frobenius algebras.

Example (Hopf algebras)

Every f.d. Hopf algebra is a Frobenius algebra.

From now on: F is a Frobenius algebra with trace tr.

Frobenius algebras: dual bases

Fix a basis B of F. The left dual basis is

$$B^\vee = \{b^\vee \mid b \in B\}$$

defined by

$$\operatorname{tr}\left(b^\vee c\right) = \delta_{b,c}, \qquad b, c \in B.$$

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

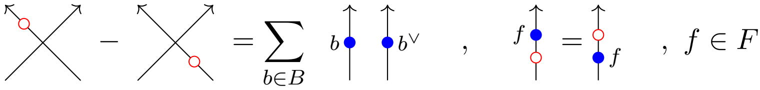
is independent of the basis B.

Affine wreath product category

Start with the wreath product category $\mathcal{W}(F)$, but add a morphism:

 $: \mathsf{Q}_+ \to \mathsf{Q}_+$

We impose the additional relations:



Call the resulting category $\mathcal{AW}(F)$ the affine wreath product category.

Now

$$\mathrm{End}_{\mathcal{AW}(F)}(\mathsf{Q}_+^{\otimes n})$$

is an affine wreath product algebra.

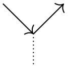
Note: $\mathcal{AW}(\Bbbk)$ is the degenerate affine Hecke category.

Adjunction

Suppose a strict monoidal category $\mathcal{C}$ has two objects Q_+ and Q_-, with

$$\mathrm{id}_{\mathsf{Q}_+} = \uparrow \quad , \quad \mathrm{id}_{\mathsf{Q}_-} = \downarrow .$$

A morphism $\mathbf{1} \to \mathsf{Q}_- \otimes \mathsf{Q}_+$ would have string diagram

 , where $= \mathrm{id}_{\mathbf{1}}$.

We typically omit the dotted line and draw:

 $: \mathbf{1} \to \mathsf{Q}_- \otimes \mathsf{Q}_+.$

Similarly, we can have

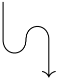
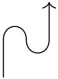
 $: \mathsf{Q}_+ \otimes \mathsf{Q}_- \to \mathbf{1}.$

Adjunction

We say that Q_- is right adjoint to Q_+ (and Q_+ is left adjoint to Q_-) if there exist morphisms

 $: \mathbf{1} \to Q_- \otimes Q_+.$ and  $: Q_+ \otimes Q_- \to \mathbf{1}.$

such that

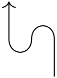
 $=$  and  $=$  .

(This is analogous to the unit-counit formulation of adjunction of functors.)

We say Q_+ and Q_- are biadjoint if they are both left and right adjoint to each other. So we also have

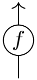
 $: \mathbf{1} \to Q_+ \otimes Q_-$ and  $: Q_- \otimes Q_+ \to \mathbf{1}$

such that

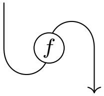
 $=$  and  $=$  .

Mates

If Q_- is right adjoint to Q_+, then every



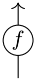
$\in \operatorname{End} \mathsf{Q}_+$ has right mate



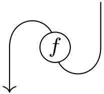
$\in \operatorname{End} \mathsf{Q}_-$.

This gives an antihomorphism $\operatorname{End} \mathsf{Q}_+ \to \operatorname{End} \mathsf{Q}_-$.

If Q_- is left adjoint to Q_+, then every



$\in \operatorname{End} \mathsf{Q}_+$ has left mate

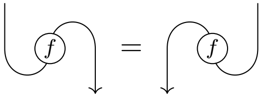


$\in \operatorname{End} \mathsf{Q}_-$.

This gives another antihomorphism $\operatorname{End} \mathsf{Q}_+ \to \operatorname{End} \mathsf{Q}_-$.

Pivotal categories

A strict monoidal category is strictly pivotal if every object has a biadjoint and right mates are always equal to left mates:



Isotopy invariance: In a strictly pivotal category, isotopic string diagrams represent the same morphism!

This allows us to use geometric intuition and topological arguments in the study of such categories.

Additive envelope

Suppose $\mathcal{C}$ is some $\Bbbk$-linear monoidal category.

Its additive envelope is the category whose:

- objects are formal finite direct sums $\bigoplus_i X_i$ of objects X_i in $\mathcal{C}$,
- morphisms

$$f\colon \bigoplus_{i=1}^{n} X_i \to \bigoplus_{j=1}^{m} Y_j$$

are $m \times n$ matrices, where the (j,i)-entry is a morphism

$$f_{i,j}\colon X_i \to Y_j.$$

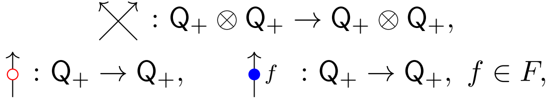
Composition is given by matrix multiplication.

The Frobenius Heisenberg category

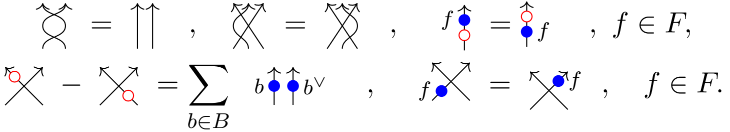
Recall the affine wreath product category $\mathcal{AW}(F)$. It is the strict $\Bbbk$-linear monoidal category with:

Objects: Generated by object Q_+.

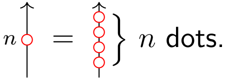
Morphisms: Generated by



with relations



For $n \in \mathbb{N}$, define

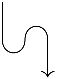
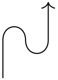


The Frobenius Heisenberg category

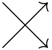
Fix a central charge $\xi \in \mathbb{Z}$, $\xi \leq 0$.

(Actually, we can take any $\xi \in \mathbb{Z}$, but we choose $\xi \leq 0$ for simplicity of exposition.)

To $\mathcal{AW}(F)$ we add another object Q_- that is right adjoint to Q_+:

 $=$  and  $=$  .

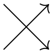
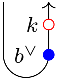
We can then define right crossings:

 $:=$  $: \mathsf{Q}_+\mathsf{Q}_- \to \mathsf{Q}_-\mathsf{Q}_+$.

(We start denoting tensor product by juxtaposition: $\mathsf{Q}_+\mathsf{Q}_- := \mathsf{Q}_+ \otimes \mathsf{Q}_-$.)

We then impose the crucial inversion relation:

The following matrix of morphisms is an isomorphism in the additive envelope:

$\Bigg[$   $,\ 0 \le k \le -\xi - 1,\ b \in B \Bigg] : \mathsf{Q}_+\mathsf{Q}_- \oplus \mathbf{1}^{\oplus(-\xi \dim F)} \to \mathsf{Q}_-\mathsf{Q}_+.$

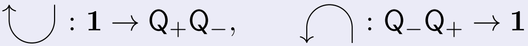
More precisely, we add in some other morphisms that are the matrix components of an inverse to the above morphism.

We call the resulting category $\mathcal{H}eis_{F,\xi}$ the Frobenius Heisenberg category.

The Frobenius Heisenberg category

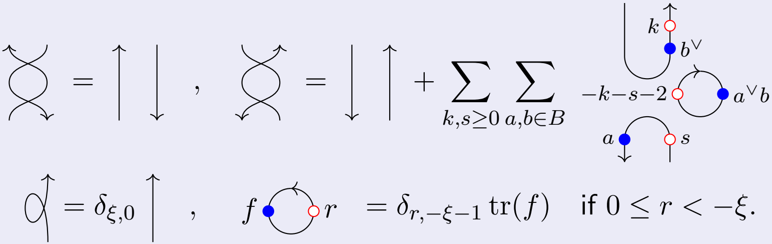
Theorem (S. 2018)

There are unique morphisms



(1)

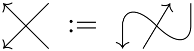
such that the following relations hold:



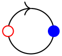
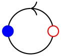
In addition $\mathcal{H}eis_{F,\xi}$ can be presented equivalently by replacing the inversion relation with the existence of morphisms (1) and above relations.

The Frobenius Heisenberg category

The previous theorem involves left crossings



and negatively dotted bubbles

$r+\xi-1$  $f \quad := (-1)^{r+1} \displaystyle\sum_{b_1,\dots,b_{r-1}\in B} \det \Big(b_{j-1}^\vee b_j$  $i-j-\xi \Big)_{i,j=1}^{r},$

if $r \leq -\xi$.

Theorem (S. 2018)

1. The objects Q_- and Q_+ are biadjoint.
2. The category $\mathcal{H}eis_{F,\xi}$ is strictly pivotal.
3. One can compute an infinite grassmannian relation, curl relations, bubble slide relations, and an alternating braid relation (omitted here).

Heisenberg categorification and actions

Action

The category $\mathcal{H}eis_{F,\xi}$ acts naturally on modules for cyclotomic affine wreath product algebras. We have a chain of algebras

$$\Bbbk = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots .$$

Then

- Q_+ acts by induction from A_n-mod to A_{n+1}-mod,
- Q_- acts by restriction from A_n-mod to A_{n-1}-mod.

The morphisms (diagrams) act by certain natural transformations.

Categorification Theorem (S. 2018)

Under a mild assumption on F, the category $\mathcal{H}eis_{F,\xi}$ categorifies the Heisenberg algebra at central charge $\xi \dim F$.

Historical remarks

Original Heisenberg category (Khovanov)

- Morphisms were planar diagrams up to isotopy, so strictly pivotal property was part of the definition.
- Central charge $\xi = -1$ and $F = \Bbbk$.

Frobenius modification (central charge -1)

- For F the zigzag algebra, defined by Cautis–Licata and studied in relation to geometry of the Hilbert scheme.
- General definition given in joint work with Rosso.
- Still have central charge $\xi = -1$.

Higher central charge (Mackaay–S.)

- Generalized to higher central charge (with $F = \Bbbk$).
- Again, pivotal property part of the definition.

Historical remarks

Inversion relation approach, $F = \Bbbk$ (Brundan)

- New approach to the definition of higher charge category (Mackaay-S.) using the inversion relation.
- Now, pivotal property is a consequence of the definition.
- Advantage: proof that category acts on modules over degenerate (cyclotomic) affine Hecke algebras is much easier. Uses a well-known Mackey-type theorem.

Current work

- Follows inversion relation approach of Brundan.
- Defines a Frobenius algebra version of higher charge category (Mackaay–S.).
- Defines a higher charge version of previous Frobenius Heisenberg category (Rosso–S.).

Historical remarks

Summarizing the relationship between the Heisenberg categories appearing in the literature, we have:

$$\begin{tikzcd} & \mathcal{H}eis_{F,\xi} \arrow[dl, dashed, "\xi=-1"'] \arrow[dr, dashed, "F=\Bbbk"] & \\ \begin{array}{c}\text{Rosso–S.}\\ \text{Cautis–Licata } (F = \text{zigzag})\end{array} \arrow[dr, dashed, "F=\Bbbk"'] & & \begin{array}{c}\text{Mackaay–S.}\\ \text{Brundan (inversion)}\end{array} \arrow[dl, dashed, "\xi=-1"] \\ & \text{Khovanov} & \end{tikzcd}$$

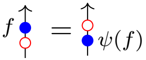
Final remarks

One can actually work in a more general setting than the one described here:

1. F can be a graded Frobenius superalgebra. Then $\mathcal{H}eis_{F,\xi}$ is a strict $\mathbb{k}$-linear graded monoidal supercategory.
2. The trace need not be symmetric. In general, there exists a Nakayama automorphism $\psi \colon F \to F$ such that

$$\mathrm{tr}(fg) = (-1)^{\bar{f}\bar{g}}\,\mathrm{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$

   Then, for instance,



$, \ f \in F,$

3. Above remarks mean we can take F to be the Clifford superalgebra. Then $\mathcal{H}eis_{F,\xi}$ acts on modules for affine Sergeev algebras (a.k.a. degenerate affine Hecke–Clifford algebras).