

Heisenberg categorification

$$\begin{array}{c} \nearrow \circ \searrow \\ \swarrow \nearrow \end{array} - \begin{array}{c} \nearrow \searrow \\ \swarrow \nearrow \circ \end{array} = \sum_{b \in B} \begin{array}{c} \uparrow \circ \\ \downarrow \circ \end{array} b^{\vee}$$

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Outline

Goals:

- 1 Explain diagrammatic categorification
- 2 Describe a family of categories that categorify the Heisenberg algebra

Overview:

- 1 Strict monoidal categories and string diagrams
- 2 Monoidally presented algebras
- 3 Adjunction and pivotal categories
- 4 The Frobenius Heisenberg category

Strict monoidal categories

A **strict monoidal category** is a category \mathcal{C} equipped with

- a bifunctor (the **tensor product**) $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, and
- a **unit object** $\mathbf{1}$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C ,
- $\mathbf{1} \otimes A = A = A \otimes \mathbf{1}$ for all objects A .

Remark: Non-strict monoidal categories

In a (not necessarily strict) **monoidal category**, the equalities above are replaced by isomorphism, and we impose some **coherence conditions**.

Every monoidal category is monoidally equivalent to a strict one.

\mathbb{k} -linear monoidal categories

Fix a commutative ground ring \mathbb{k} .

A **strict \mathbb{k} -linear monoidal category** is a strict monoidal category such that

- each morphism space is a \mathbb{k} -module,
- composition of morphisms is \mathbb{k} -bilinear,
- tensor product of morphisms is \mathbb{k} -bilinear.

The interchange law

The axioms of a strict monoidal category imply the **interchange law**: For $A_1 \xrightarrow{f} A_2$ and $B_1 \xrightarrow{g} B_2$, the following diagram commutes:

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{1 \otimes g} & A_1 \otimes B_2 \\ f \otimes 1 \downarrow & \searrow f \otimes g & \downarrow f \otimes 1 \\ A_2 \otimes B_1 & \xrightarrow{1 \otimes g} & A_2 \otimes B_2 \end{array}$$

Strict monoidal categories

Example (Monoids)

A (strict) monoidal category with one object is simply a commutative monoid. More precisely, the endomorphisms of $\mathbf{1}$ form a commutative monoid.

Conversely, every commutative monoid gives rise to a one-object monoidal category.

Example (Associative algebras)

A (strict) \mathbb{k} -linear monoidal category is simply a commutative associative unital \mathbb{k} -algebra.

Categorification via split Grothendieck group

Suppose \mathcal{C} is an additive category (i.e. have \oplus).

$\text{Iso}_{\mathbb{Z}}(\mathcal{C}) =$ free abelian group generated by isom. classes of objects in \mathcal{C} .

The **split Grothendieck group** of \mathcal{C} is

$$K_0(\mathcal{C}) = \text{Iso}_{\mathbb{Z}}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$

If \mathcal{C} is **monoidal**, then $K_0(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to **categorify** a ring R is to find an additive monoidal category \mathcal{C} such that

$$K_0(\mathcal{C}) \cong R \quad \text{as rings.}$$

The Heisenberg algebra

Let \mathfrak{h} be the infinite-dimensional Heisenberg Lie algebra.

Thus, \mathfrak{h} is the complex Lie algebra with basis

$$\{c, q_n^\pm : n \geq 1\}$$

and product

$$[q_m^+, q_n^+] = [q_m^-, q_n^-] = [c, q_n^\pm] = 0, \quad [q_m^+, q_n^-] = \delta_{m,n} n c.$$

The associative Heisenberg algebra at **central charge** $\xi \in \mathbb{Z}$ is

$$U(\mathfrak{h}) / \langle c - \xi \rangle.$$

We will describe categories that categorify these algebras.

String diagrams

Let's draw pictures! Fix a strict monoidal category \mathcal{C} .

We will denote a morphism $f: A \rightarrow B$ by:



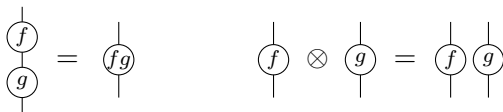
The **identity map** $1_A: A \rightarrow A$ is a string with no label:



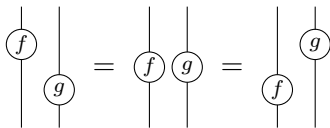
We sometimes omit the object labels when they are clear or unimportant.

String diagrams

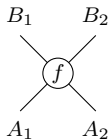
Composition is **vertical stacking** and tensor product is **horizontal juxtaposition**:



The **interchange law** then becomes:



A morphism $f: A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ can be depicted:



Presentations of strict monoidal categories

One can give **presentations** of some strict \mathbb{k} -linear monoidal categories, just as for monoids, groups, algebras, etc.

Objects: If the objects are generated by some collection $A_i, i \in I$, then we have all possible tensor products of these objects:

$$\mathbf{1}, \quad A_i, \quad A_i \otimes A_j \otimes A_k \otimes A_\ell, \quad \text{etc.}$$

Morphisms: If the morphisms are generated by some collection $f_j, j \in J$, then we have all possible compositions and tensor products of these morphisms (whenever these make sense):

$$1_{A_i}, \quad f_j \otimes (f_i f_k) \otimes (f_\ell), \quad \text{etc.}$$

We then often impose some **relations** on these morphism spaces.

String diagrams: We can build complex diagrams out of our simple generating diagrams.

Monoidally generated symmetric groups

Define a strict monoidal category \mathcal{S} with one generating object Q_+ and denote

$$1_{Q_+} = \uparrow$$

We have one generating morphism

$$\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} : Q_+ \otimes Q_+ \rightarrow Q_+ \otimes Q_+.$$

We impose the relations:

$$\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array}.$$

It is straightforward to verify that

$$\text{End}_{\mathcal{S}}(Q_+^{\otimes n}) = S_n$$

is the **symmetric group** on n letters.

Monoidally generated symmetric groups

This monoidal presentation of S_n is very efficient! We only needed

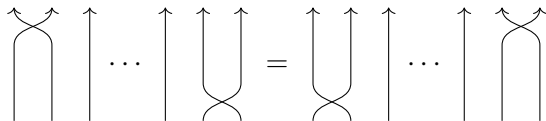
- one generating morphism, and
- two relations,

to get **all** the symmetric groups.

Note that the “distant braid relation”

$$s_i s_j = s_j s_i, \quad |i - j| > 1$$

for simple transpositions follows for free from the interchange law:



Note: If we define \mathcal{S} to be \mathbb{k} -linear, then $\text{End}_{\mathcal{S}}(\mathbb{Q}_+^{\otimes n}) = \mathbb{k}S_n$.

The degenerate affine Hecke category

Start again with the strict \mathbb{k} -linear monoidal category \mathcal{S} , but add a morphism:

$$\uparrow_{\circ} : \mathbb{Q}_+ \rightarrow \mathbb{Q}_+$$

We impose the additional relation:

$$\begin{array}{c} \nearrow \\ \circ \\ \searrow \\ \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \\ \circ \\ \nearrow \\ \searrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} .$$

Now

$$\text{End}(\mathbb{Q}_+^{\otimes n})$$

is the **degenerate affine Hecke algebra** (of type A).

The wreath product category

Fix an associative \mathbb{k} -algebra F . We add an endomorphism of \mathbb{Q}_+ for each element of F .

More precisely, let $\mathcal{W}(F)$ be the strict \mathbb{k} -linear monoidal category obtained from \mathcal{S} by adding morphisms such that we have an algebra homomorphism:

$$F \rightarrow \text{End } \mathbb{Q}_+, \quad f \mapsto \begin{array}{c} \uparrow \\ \bullet \\ f \end{array}$$

We impose the additional relations:

$$\begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array}, \quad f \in F$$

Example of diagrammatic proof:

$$\begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} \implies \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} \implies \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \bullet \\ f \\ \searrow \end{array}$$

The wreath product category

$$\text{End}_{\mathcal{W}(F)}(\mathbb{Q}_+^{\otimes n}) = F^{\otimes n} \rtimes S_n$$

is a wreath product algebra.

As a vector space,

$$F^{\otimes n} \rtimes S_n = F^{\otimes n} \otimes_{\mathbb{k}} \mathbb{k}S_n.$$

Multiplication is determined by

$$(f_1 \otimes \pi_1)(f_2 \otimes \pi_2) = f_1(\pi_1 \cdot f_2) \otimes \pi_1\pi_2, \quad f_1, f_2 \in F^{\otimes n}, \pi_1, \pi_2 \in S_n,$$

where $\pi_1 \cdot f_2$ denotes the natural action of S_n on $F^{\otimes n}$ by permutation of the factors.

Note: $\mathcal{W}(\mathbb{k}) = \mathcal{S}$, the symmetric group category.

Want: An affine version of the wreath product category. $F = \mathbb{k}$ should recover the degenerate affine Hecke category.

Frobenius algebras: Definition

Frobenius algebra

A **Frobenius algebra** is a f.d. associative algebra F together with a linear trace map

$$\mathrm{tr}: F \rightarrow \mathbb{k}$$

such that the induced map

$$F \rightarrow \mathrm{Hom}_{\mathbb{k}}(F, \mathbb{k}), \quad f \mapsto (g \mapsto \mathrm{tr}(gf)),$$

is an isomorphism.

For simplicity, we assume that the trace is symmetric:

$$\mathrm{tr}(fg) = \mathrm{tr}(gf), \quad \text{for all } f, g \in F.$$

Frobenius algebras: Examples

Example (\mathbb{k})

\mathbb{k} is a Frobenius algebra with $\text{tr} = 1_{\mathbb{k}}$.

Example (Matrix algebra)

Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example ($\mathbb{k}[x]/(x^k)$)

$\mathbb{k}[x]/(x^k)$ is a Frobenius algebra with

$$\text{tr}(x^\ell) = \delta_{\ell, k-1}.$$

Frobenius algebras: Examples

Example (Group algebra)

Suppose G is a finite group.

The **group algebra** $\mathbb{k}G$ is a Frobenius algebra with

$$\mathrm{tr}(g) = \delta_{g,1_G}, \quad g \in G.$$

Example (Zigzag algebra)

Associated to every quiver is a **zigzag algebra**. These are Frobenius algebras.

Example (Hopf algebras)

Every f.d. Hopf algebra is a Frobenius algebra.

From now on: F is a Frobenius algebra with trace tr .

Frobenius algebras: dual bases

Fix a basis B of F . The **left dual basis** is

$$B^\vee = \{b^\vee \mid b \in B\}$$

defined by

$$\mathrm{tr}(b^\vee c) = \delta_{b,c}, \quad b, c \in B.$$

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

is independent of the basis B .

Affine wreath product category

Start with the wreath product category $\mathcal{W}(F)$, but add a morphism:

$$\uparrow \circ : \mathbb{Q}_+ \rightarrow \mathbb{Q}_+$$

We impose the additional relations:

$$\begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \\ \circ \end{array} = \sum_{b \in B} \begin{array}{c} \uparrow \\ \bullet \\ | \\ \bullet \\ \uparrow \end{array} b^{\vee} \quad , \quad \begin{array}{c} \uparrow \\ \bullet \\ | \\ \circ \\ \uparrow \end{array} f = \begin{array}{c} \uparrow \\ \circ \\ | \\ \bullet \\ \uparrow \end{array} f \quad , \quad f \in F$$

Call the resulting category $\mathcal{AW}(F)$ the **affine wreath product category**.

Now

$$\text{End}_{\mathcal{AW}(F)}(\mathbb{Q}_+^{\otimes n})$$

is an **affine wreath product algebra**.

Note: $\mathcal{AW}(\mathbb{k})$ is the degenerate affine Hecke category.

Adjunction

Suppose a strict monoidal category \mathcal{C} has two objects Q_+ and Q_- , with

$$1_{Q_+} = \uparrow \quad , \quad 1_{Q_-} = \downarrow .$$

A morphism $\mathbf{1} \rightarrow Q_- \otimes Q_+$ would have string diagram

$$\begin{array}{c} \diagup \quad \diagdown \\ \downarrow \\ \cdots \end{array} \quad , \quad \text{where} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} = 1_{\mathbf{1}} .$$

We typically omit the dotted line and draw:

$$\begin{array}{c} \curvearrowright \\ \uparrow \end{array} : \mathbf{1} \rightarrow Q_- \otimes Q_+ .$$

Similarly, we can have

$$\begin{array}{c} \curvearrowleft \\ \downarrow \end{array} : Q_+ \otimes Q_- \rightarrow \mathbf{1} .$$

Adjunction

We say that Q_- is **right adjoint** to Q_+ (and Q_+ is **left adjoint** to Q_-) if there exist morphisms

$$\cup : \mathbf{1} \rightarrow Q_- \otimes Q_+ \quad \text{and} \quad \cap : Q_+ \otimes Q_- \rightarrow \mathbf{1}.$$

such that

$$\downarrow \cup = \downarrow \quad \text{and} \quad \cap \uparrow = \uparrow.$$

(This is analogous to the unit-counit formulation of adjunction of functors.)

We say Q_+ and Q_- are **biadjoint** if they are both left and right adjoint to each other. So we also have

$$\cup : \mathbf{1} \rightarrow Q_+ \otimes Q_- \quad \text{and} \quad \cap : Q_- \otimes Q_+ \rightarrow \mathbf{1}$$

such that

$$\cup \downarrow = \downarrow \quad \text{and} \quad \cap \uparrow = \uparrow.$$

Mates

If Q_- is **right adjoint** to Q_+ , then every

$$\begin{array}{c} \uparrow \\ \textcircled{f} \\ \downarrow \end{array} \in \text{End } Q_+ \quad \text{has right mate} \quad \begin{array}{c} \downarrow \\ \textcircled{f} \\ \uparrow \end{array} \in \text{End } Q_-.$$

This gives an antihomomorphism $\text{End } Q_+ \rightarrow \text{End } Q_-$.

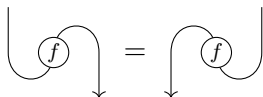
If Q_- is **left adjoint** to Q_+ , then every

$$\begin{array}{c} \uparrow \\ \textcircled{f} \\ \downarrow \end{array} \in \text{End } Q_+ \quad \text{has left mate} \quad \begin{array}{c} \downarrow \\ \textcircled{f} \\ \uparrow \end{array} \in \text{End } Q_-.$$

This gives another antihomomorphism $\text{End } Q_+ \rightarrow \text{End } Q_-$.

Pivotal categories

A strict monoidal category is **strictly pivotal** if every object has a biadjoint and right mates are always equal to left mates:



Isotopy invariance: In a strictly pivotal category, isotopic string diagrams represent the same morphism!

This allows us to use geometric intuition and topological arguments in the study of such categories.

Additive envelope

Suppose \mathcal{C} is some \mathbb{k} -linear monoidal category.

Its **additive envelope** is the category whose:

- **objects** are formal finite direct sums $\bigoplus_i X_i$ of objects X_i in \mathcal{C} ,
- **morphisms**

$$f: \bigoplus_{i=1}^n X_i \rightarrow \bigoplus_{j=1}^m Y_j$$

are $m \times n$ matrices, where the (j, i) -entry is a morphism

$$f_{i,j}: X_i \rightarrow Y_j.$$

Composition is given by matrix multiplication.

The Frobenius Heisenberg category

Recall the affine wreath product category $\mathcal{AW}(F)$. It is the strict \mathbb{k} -linear monoidal category with:

Objects: Generated by object Q_+ .

Morphisms: Generated by

$$\begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} : Q_+ \otimes Q_+ \rightarrow Q_+ \otimes Q_+,$$

$$\begin{array}{c} \uparrow \\ \circ \\ | \end{array} : Q_+ \rightarrow Q_+, \quad \begin{array}{c} \uparrow \\ \bullet \\ | \end{array} f : Q_+ \rightarrow Q_+, \quad f \in F,$$

with relations

$$\begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, \quad \begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array}, \quad \begin{array}{c} \uparrow \\ \bullet \\ \circ \\ | \end{array} f = \begin{array}{c} \uparrow \\ \circ \\ \bullet \\ | \end{array} f, \quad f \in F,$$

$$\begin{array}{c} \uparrow \\ \circ \\ \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nearrow \searrow \\ \times \\ \circ \\ | \end{array} = \sum_{b \in B} \begin{array}{c} \uparrow \\ \bullet \\ \uparrow \\ \bullet \\ | \end{array} b^\vee, \quad \begin{array}{c} \uparrow \\ \bullet \\ \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} f = \begin{array}{c} \nearrow \searrow \\ \times \\ \bullet \\ | \end{array} f, \quad f \in F.$$

For $n \in \mathbb{N}$, define

$$\begin{array}{c} \uparrow \\ \circ \\ | \end{array} = \left. \begin{array}{c} \uparrow \\ \circ \\ \circ \\ \circ \\ | \end{array} \right\} n \text{ dots.}$$

The Frobenius Heisenberg category

Fix a **central charge** $\xi \in \mathbb{Z}$, $\xi \leq 0$.

(Actually, we can take any $\xi \in \mathbb{Z}$, but we choose $\xi \leq 0$ for simplicity of exposition.)

To $\mathcal{AW}(F)$ we add another object Q_- that is **right adjoint** to Q_+ :

$$\begin{array}{c} \text{L-shaped curve} \\ \downarrow \end{array} = \downarrow \quad \text{and} \quad \begin{array}{c} \text{R-shaped curve} \\ \uparrow \end{array} = \uparrow.$$

We can then define **right crossings**:

$$\begin{array}{c} \text{Crossing} \\ \times \end{array} := \begin{array}{c} \text{Right crossing} \\ \curvearrowright \end{array} : Q_+ Q_- \rightarrow Q_- Q_+.$$

(We start denoting tensor product by juxtaposition: $Q_+ Q_- := Q_+ \otimes Q_-$.)

The Frobenius Heisenberg category

We then impose the crucial **inversion relation**:

The following matrix of morphisms is an isomorphism in the additive envelope:

$$\left[\begin{array}{c} \text{X} \\ \text{U} \end{array}, 0 \leq k \leq -\xi - 1, b \in B \right] : \mathbf{Q}_+ \mathbf{Q}_- \oplus \mathbf{1}^{\oplus(-\xi \dim F)} \rightarrow \mathbf{Q}_- \mathbf{Q}_+.$$

More precisely, we add in some other morphisms that are the matrix components of an inverse to the above morphism.

We call the resulting category $\mathcal{H}eis_{F,\xi}$ the **Frobenius Heisenberg category**.

The Frobenius Heisenberg category

Theorem (S. 2018)

There are unique morphisms

$$\uparrow \cup : \mathbf{1} \rightarrow \mathbf{Q}_+ \mathbf{Q}_-, \quad \downarrow \cap : \mathbf{Q}_- \mathbf{Q}_+ \rightarrow \mathbf{1} \quad (1)$$

such that the following relations hold:

$$\begin{aligned} \text{crossing} &= \uparrow \downarrow, & \text{crossing} &= \downarrow \uparrow + \sum_{k,s \geq 0} \sum_{a,b \in B} \text{diagram} \\ \text{loop} &= \delta_{\xi,0} \uparrow, & \text{loop} &= \delta_{r,-\xi-1} \text{tr}(f) \text{ if } 0 \leq r < -\xi. \end{aligned}$$

In addition $\mathcal{H}eis_{F,\xi}$ can be presented equivalently by replacing the inversion relation with the existence of morphisms (1) and above relations.

The Frobenius Heisenberg category

The previous theorem involves **left crossings**

$$\begin{array}{c} \nearrow \\ \searrow \\ \swarrow \\ \nwarrow \end{array} := \begin{array}{c} \curvearrowright \\ \nearrow \\ \nwarrow \end{array}$$

and **negatively dotted** bubbles

$$r+\xi-1 \begin{array}{c} \curvearrowright \\ \circ \\ \bullet \\ \circ \end{array} f := (-1)^{r+1} \sum_{b_1, \dots, b_{r-1} \in B} \det \left(b_{j-1}^\vee b_j \begin{array}{c} \curvearrowright \\ \circ \\ \bullet \\ \circ \end{array} i-j-\xi \right)_{i,j=1}^r,$$

if $r \leq -\xi$.

Theorem (S. 2018)

- 1 The objects Q_- and Q_+ are **biadjoint**.
- 2 The category $\mathcal{H}eis_{F,\xi}$ is **strictly pivotal**.
- 3 One can compute an infinite grassmannian relation, curl relations, bubble slide relations, and an alternating braid relation (omitted here).

Heisenberg categorification and actions

Action

The category $\mathcal{H}eis_{F,\xi}$ acts naturally on modules for **cyclotomic wreath product algebras**. We have a chain of algebras

$$\mathbb{k} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots .$$

Then

- Q_+ acts by induction from $A_n\text{-mod}$ to $A_{n+1}\text{-mod}$,
- Q_- acts by restriction from $A_n\text{-mod}$ to $A_{n-1}\text{-mod}$.

The morphisms (diagrams) act by certain natural transformations.

Categorification Theorem (S. 2018)

Under a mild assumption on F , the category $\mathcal{H}eis_{F,\xi}$ **categorifies** the Heisenberg algebra at central charge $\xi \dim F$.

Historical remarks

Original Heisenberg category (Khovanov)

- Morphisms were planar diagrams **up to isotopy**, so strictly pivotal property was part of the definition.
- Central charge $\xi = -1$ and $F = \mathbb{k}$.

Frobenius modification (central charge -1)

- For F the zigzag algebra, defined by Cautis–Licata and studied in relation to geometry of the **Hilbert scheme**.
- General definition given in joint work with Rosso.
- Still have central charge $\xi = -1$.

Higher central charge (Mackaay–S.)

- Generalized to higher central charge (with $F = \mathbb{k}$).
- Again, pivotal property part of the definition.

Historical remarks

Inversion relation approach, $F = \mathbb{k}$ (Brundan)

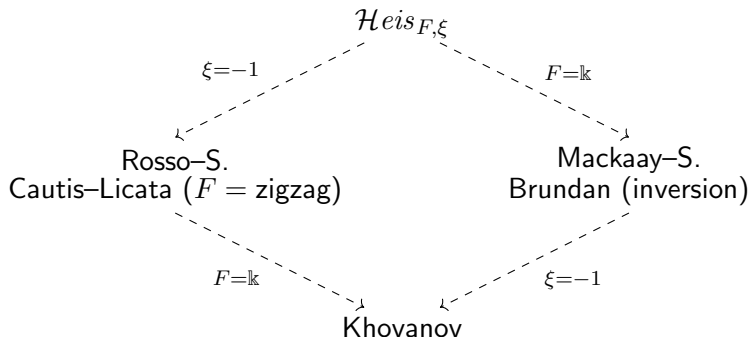
- New approach to the definition of higher charge category (Mackaay-S.) using the inversion relation.
- Now, pivotal property is a **consequence** of the definition.
- **Advantage**: proof that category acts on modules over degenerate (cyclotomic) affine Hecke algebras is much easier. Uses a well-known Mackey-type theorem.

Current work

- Follows inversion relation approach of Brundan.
- Defines a Frobenius algebra version of higher charge category (Mackaay-S.).
- Defines a higher charge version of previous Frobenius Heisenberg category (Rosso-S.).

Historical remarks

Summarizing the relationship between the Heisenberg categories appearing in the literature, we have:



Final remarks

One can actually work in a more general setting than the one described here:

- 1 F can be a **graded Frobenius superalgebra**. Then $\mathcal{H}eis_{F,\xi}$ is a strict \mathbb{k} -linear **graded monoidal supercategory**.
- 2 The trace need not be symmetric. In general, there exists a **Nakayama automorphism** $\psi: F \rightarrow F$ such that

$$\mathrm{tr}(fg) = (-1)^{\bar{f}\bar{g}} \mathrm{tr}(g\psi(f)) \quad \text{for all } f, g \in F.$$

Then, for instance,

$$f \begin{array}{c} \uparrow \\ \bullet \\ \circ \end{array} = \begin{array}{c} \uparrow \\ \circ \\ \bullet \end{array} \psi(f), \quad f \in F,$$

- 3 Above remarks mean we can take F to be the **Clifford superalgebra**. Then $\mathcal{H}eis_{F,\xi}$ acts on modules for **affine Sergeev algebras** (a.k.a. **degenerate affine Hecke–Clifford algebras**).