Adjunction

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Dual spaces

Suppose V and W are finite-dimensional complex vector spaces.

The dual of V is the space of linear functionals on V:

$$V^* := \{f : V \to \mathbb{C} : f \text{ is linear}\}.$$

It is itself a vector space under

- pointwise addition:

$$(f + g)(v) := f(v) + g(v), \quad f, g \in V^*, \ v \in V,$$

- scalar multiplication:

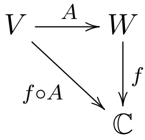
$$(\alpha f)(v) := \alpha f(v), \quad \alpha \in \mathbb{C}, \ f \in V^*, \ v \in V.$$

Transposes

Suppose we have a linear map

$$A\colon V \to W.$$

Then, given $f \in W^*$, we can produce an element of V^*:



The transpose of $A\colon V \to W$ is the linear map

$$A^T\colon W^* \to V^*,$$
$$A^T f := f \circ A, \quad f \in W^*.$$

Inner product spaces

An inner product on V is a map

$$\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{C},$$

satisfying the following axioms for all $u, v, w \in V$ and all $\alpha \in \mathbb{C}$:

- conjugate symmetry:
$$\langle u, v \rangle_V = \overline{\langle v, u \rangle_V}$$

- linearity in the first argument:
$$\langle \alpha v, w \rangle_V = \alpha \langle v, w \rangle_V$$
$$\langle u + v, w \rangle_V = \langle u, w \rangle_V + \langle v, w \rangle_V$$

- positive-definiteness:
$$\langle v, v \rangle_V \geq 0$$
$$\langle v, v \rangle_V = 0 \iff v = 0$$

Inner product spaces

Example

If $V = \mathbb{C}^n$ (column vectors), then we have the standard inner product

$$\langle u, v \rangle = u^T \bar{v}.$$

Suppose V has an inner product. For each $v \in V$, we can define a linear map

$$f_v \colon V \to C, \quad f_v(u) = \langle u, v \rangle_V, \quad u \in V.$$

So $f_v \in V^*$.

One can check that this defines an isomorphism of vector spaces

$$V \cong V^*, \quad v \mapsto f_v.$$

Summary: Inner products allow us to identify a f.d. vector space with its dual.

Adjoints

Suppose V and W both have inner products, and that we have a linear map

$$A \colon V \to W.$$

Then we have the following:

$$\begin{array}{ccc} W^* & \xrightarrow{A^T} & V^* \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ W & \xrightarrow[A^*]{} & V \end{array}$$

We can complete the square to a commutative diagram. The bottom map is the adjoint A^* of A.

The adjoint is uniquely determined by the fact that

$$\langle Av, w \rangle_W = \langle v, A^* w \rangle_V, \quad v \in V,\ w \in W.$$

Adjoints

Example

Suppose $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, with the standard inner products.

Every linear map $A\colon V \to W$ corresponds to a matrix.

The defining property of the adjoint is that

$$(Av)^T \bar{w} = \langle Av, w \rangle = \langle v, A^* w \rangle = v^T \overline{A^* w}, \quad v \in V,\ w \in W.$$

So, for all $v \in V$ and $w \in W$, we have

$$v^T A^T \bar{w} = v^T \overline{A^* w} = v^T \overline{A^*} \bar{w}.$$

This implies that $A^T = \overline{A^*}$, which is equivalent to

$$A^* = \overline{A^T}.$$

So the adjoint A^* is the conjugate transpose of the matrix A.

Categories (Definition)

A category $\mathcal{C}$ consists of

- a class of objects $\operatorname{Ob} \mathcal{C}$,
- a class of morphisms $\hom_{\mathcal{C}}(X, Y)$ for all $X, Y \in \operatorname{Ob} \mathcal{C}$,

together with a composition

$$\hom_{\mathcal{C}}(Y, Z) \times \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto f \circ g,$$

and an identity morphism $1_X \in \hom_{\mathcal{C}}(X, X)$ for all objects $X \in \operatorname{Ob} \mathcal{C}$.

The composition must be associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever $f \circ g$ and $g \circ h$ are defined.

The identity morphism has the property that

$$1_Y \circ f = f = f \circ 1_X \quad \text{for all } f \in \hom_{\mathcal{C}}(X, Y).$$

Categories (Examples)

Example (Sets)

- Objects: sets
- Morphisms: set maps

Example (Vector spaces)

- Objects: vector spaces over a fixed field
- Morphisms: linear maps

Example (Groups)

- Objects: groups
- Morphisms: group homomorphisms

Categories (Examples)

Example (Rings)

- Objects: rings
- Morphisms: ring homomorphisms

Example (Topological spaces)

- Objects: topological spaces
- Morphisms: continuous maps

Other examples

- modules over a fixed ring
- smooth manifolds
- algebraic varieties
- ...

Functors

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories.

A functor F from $\mathcal{C}$ to $\mathcal{D}$ consists of

- a map $F\colon \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$,
- for all $X, Y \in \operatorname{Ob}\mathcal{C}$, a map $F\colon \hom_{\mathcal{C}}(X,Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$.

We require that the map on morphisms respects composition:

$$F(f \circ g) = F(f) \circ F(g),$$

whenever the composition of $f, g \in \hom \mathcal{C}$ is defined.

We also require it to preserve identities:

$$F(1_X) = 1_{F(X)}.$$

Functors

Example (Forgetful functors)

We can define a functor

$$F\colon \text{category of groups} \to \text{category of sets}$$

as follows:

- for a group G, we define $F(G)$ to be the underlying set of G,
- for a group homomorphism $f\colon G_1 \to G_2$, we define $F(f)$ to be the underlying set map.

So F just forgets the group structure.

There are many other examples of forgetful functors:

- category of rings $\to$ category of abelian groups, $(R,+,\cdot) \mapsto (R,+)$
- category of rings $\to$ category of sets
- category of vector spaces $\to$ category of sets

Functors

Example (Double dual)

There is a functor from the category of complex vector spaces to itself that

- maps any vector space to its double dual (the dual of its dual space),
- maps any linear map to its double transpose.

Example (Fundamental group)

Suppose

- **Top** is the category of pointed topological spaces (topological spaces together with a distinguished point),
- **Group** is the category of groups.

We have a functor **Top** $\to$ **Group** that maps a pointed topological space to its fundamental group.

Adjoint functors: general idea

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. Let **Set** be the category of sets.

General philosophy

We will think of

$$\hom_{\mathcal{C}}(\cdot,\cdot)\colon \mathcal{C}\times\mathcal{C}\to \textbf{Set}$$

as a categorical analogue of an inner product on the category $\mathcal{C}$.

Goal

Define an appropriate notion of adjoint functor.

First guess

The adjoint of a functor $F\colon \mathcal{C}\to\mathcal{D}$ is a functor $G\colon \mathcal{D}\to\mathcal{C}$ such that

$$\hom_{\mathcal{D}}(FX,Y)=\hom_{\mathcal{C}}(X,GY),\quad X\in \operatorname{Ob}\mathcal{C},\ Y\in\operatorname{Ob}\mathcal{D}.$$

Problem 1: Lack of symmetry

For inner products, we had conjugate symmetry:

$$\langle u, v \rangle_V = \overline{\langle v, u \rangle_V}, \quad u, v \in V.$$

It follows that $(A^*)^* = A$.

There is no such symmetry (in general) in the setting of categories.

Solution: Break the symmetry!

Second guess

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G \colon \mathcal{D} \to \mathcal{C}$ (and G is right adjoint to F) if

$$\hom_{\mathcal{D}}(FX, Y) = \hom_{\mathcal{C}}(X, GY), \quad X \in \operatorname{Ob} \mathcal{C},\ Y \in \operatorname{Ob} \mathcal{D}.$$

Problem 2: Equality is weird

Example

Define two groups as follows:

$$G = \{a, b\}, \quad a^2 = b^2 = 1,\ ab = ba = b,$$
$$H = \{x, y\}, \quad x^2 = y^2 = 1,\ xy = yx = y.$$

Are these the same group (i.e. are they equal)? Technically, no.

However, they are isomorphic via the map $a \mapsto x$, $b \mapsto y$.

In fact, both are isomorphic to the cyclic group of order 2.

Observation: In category theory, the concept of isomorphism is more fundamental than the notion of equality.

From equality to isomorphism

Our observation suggests that the condition

$$\hom_C(FX, Y) = \hom_D(X, GY), \quad X \in \operatorname{Ob} \mathcal{C},\ Y \in \operatorname{Ob} \mathcal{D},$$

is not very natural.

Solution: Change from equality to isomorphism!

Third guess

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G \colon \mathcal{D} \to \mathcal{C}$ (and G is right adjoint to F) if

$$\hom_{\mathcal{D}}(FX, Y) \cong \hom_{\mathcal{C}}(X, GY), \quad X \in \operatorname{Ob} \mathcal{C},\ Y \in \operatorname{Ob} \mathcal{D}.$$

(Here $\cong$ is isomorphism of sets.)

Problem 3: Naturality

Generally, in mathematics, natural constructions should respect the structure that is present:

1. vector space homomorphisms should be linear,
2. group homomorphisms should commute with the group operation,
3. ring homomorphisms should commute with multiplication and addition.

Our current definition doesn't really fully respect all of the structure of the categories $\mathcal{C}$ and $\mathcal{D}$ coming from morphisms.

Suppose

$$f \in \hom_{\mathcal{C}}(X_1, X_2), \quad g \in \hom_{\mathcal{C}}(Y_1, Y_2).$$

Then we have a map

$$\mu_{f,g} \colon \hom_{\mathcal{C}}(X_2, Y_1) \to \hom_{\mathcal{C}}(X_1, Y_2),$$
$$A \mapsto g \circ A \circ f.$$

Naturality

Suppose

$$\hom_C(FX, Y) \cong \hom_D(X, GY), \quad X \in \operatorname{Ob}\mathcal{C},\ Y \in \operatorname{Ob}\mathcal{D}.$$

We say these isomorphisms are natural if the diagram

$$\begin{array}{ccc}
\hom_{\mathcal{C}}(FX_2, Y_1) & \xrightarrow{\cong} & \hom_{\mathcal{D}}(X_2, GY_1) \\
{\scriptstyle \mu_{F(f),g}} \big\downarrow & & \big\downarrow {\scriptstyle \mu_{f,G(g)}} \\
\hom_{\mathcal{C}}(FX_1, Y_2) & \xrightarrow{\cong} & \hom_{\mathcal{D}}(X_1, GY_2)
\end{array}$$

commutes for all

$$f \in \hom_{\mathcal{C}}(X_1, X_2), \quad g \in \hom_{\mathcal{D}}(Y_1, Y_2).$$

Adjoint functors: final definition

Definition (Adjoint functors)

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G \colon \mathcal{D} \to \mathcal{C}$ (and G is right adjoint to F) if we have natural isomorphisms

$$\hom_{\mathcal{D}}(FX, Y) \cong \hom_{\mathcal{C}}(X, GY), \quad X \in \operatorname{Ob} \mathcal{C},\ Y \in \operatorname{Ob} \mathcal{D}.$$

Now that we've formulated the correct notion of adjoint functors, there is an obvious question:

Question: Should we care? Are there nice/familiar examples of adjoint functors?

Answer: YES!

Application: linear algebra

Let **Vect** denote the category of complex vector spaces.

Recall that we have the forgetful functor

$$\textbf{forget}\colon \textbf{Vect} \to \textbf{Set}.$$

Let's try to find a left adjoint to **forget**.

So we want a functor $F\colon \textbf{Set} \to \textbf{Vect}$ such that we have natural isomorphisms

$$\hom_{\textbf{Vect}}(FX, Y) \cong \hom_{\textbf{Set}}(X, \textbf{forget}(Y)),$$

for X a set and Y a vector space.

So, if X is a set and Y is a vector space, we want to associate to every set map $f\colon X \to Y$ a vector space map $Ff\colon FX \to Y$.

Application: linear algebra

Recall from linear algebra

Suppose V and W are vector spaces. If

- v_i, $i \in I$, is a basis of V, and
- w_i, $i \in I$, are arbitrary elements of W,

then there is a unique linear map

$$A \colon V \to W \quad \text{such that} \quad Av_i = w_i.$$

Put another way: If

- B is a basis of V, and
- $f \colon B \to W$ is a map of sets,

then there is a unique way to extend f to a linear map

$$A \colon V \to W \quad \text{such that} \quad Av = fv \ \forall \ v \in B.$$

Application: linear algebra

Recall: We want a functor $F\colon \textbf{Set} \to \textbf{Vect}$ such that we have natural isomorphisms

$$\hom_{\textbf{Vect}}(FX, Y) \cong \hom_{\textbf{Set}}(X, \textbf{forget}(Y)),$$

for X a set and Y a vector space.

So FX should be a vector space with basis X.

Free vector spaces

Question: Given a set X, how do we form a vector space with basis X?

For a set X, let

$$FX = \{f\colon X \to \mathbb{C} : f(x) = 0 \text{ for all but finitely many } x \in X\}.$$

For $x \in X$, we have the Dirac delta function:

$$\delta_x\colon X \to \mathbb{C}, \quad \delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Exercise: The δ_x, $x \in X$, are a basis for FX.

So if we identify x with δ_x, then FX is a vector space with basis X.

FX is called the free vector space on the set X.

Example of adjoint functors in linear algebra

To every set map $f\colon X \to Y$, we can associate a natural linear map $Ff\colon FX \to FY$, and show that we have a functor **Set** $\to$ **Vect**.

Theorem

The free vector space functor

$$F\colon \textbf{Set} \to \textbf{Vect}$$

is left adjoint to the forgetful functor

$$\textbf{forget}\colon \textbf{Vect} \to \textbf{Set}.$$

We've done most of the work in proving this. One just needs to verify that the isomorphisms

$$\hom_{\textbf{Vect}}(FX, Y) \cong \hom_{\textbf{Set}}(X, \textbf{forget}(Y)),$$

are natural (exercise).

Application: group theory

Let **Group** be the category of groups.

We have a forgetful functor

$$\mathbf{forget}\colon \mathbf{Group} \to \mathbf{Set}.$$

Just as for vector spaces, its left adjoint is the free group functor.

Suppose X is a set. The free group on X is the group FX with elements that are words in the alphabet X and formal inverses of elements of X:

$$a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}, \quad a_1, a_2, \ldots, a_n \in X.$$

Multiplication is given by concatenation and cancelling inverses: e.g.

$$(aba^{-1}bc^{-1}ab)(b^{-1}ac) = aba^{-1}bc^{-1}aac.$$

Application: group theory

If $f\colon X \to Y$ is a map of sets, then we have the corresponding map of free groups:

$$FX \to FY, \quad a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1} \mapsto f(a_1)^{\pm 1} f(a_2)^{\pm 1} \cdots f(a_n)^{\pm 1}.$$

So we have a free group functor

$$F\colon \mathbf{Set} \to \mathbf{Group}.$$

Theorem

The free group functor

$$F\colon \mathbf{Set} \to \mathbf{Group}$$

is left adjoint to the forgetful functor

$$\mathbf{forget}\colon \mathbf{Group} \to \mathbf{Set}.$$

Note: Can replace **Group** by the category of abelian groups to get the free abelian group on a set.

Application: adjoining an identity to a ring

Let **Ring** be the category of rings.

Let **Rng** be the category of general rings (rings without the multiplicative identity axiom).

The left adjoint to the forgetful functor

$$\mathbf{forget}\colon \mathbf{Ring} \to \mathbf{Rng}$$

is a functor that formally adjoins an identity to a ring. It maps a general ring R to $R \times \mathbb{Z}$ with multiplication determined by

$$(r,0)(0,1) = (r,0) = (0,1)(r,0),$$
$$(r,0)(s,0) = (rs,0),$$
$$(0,1)(0,1) = (0,1).$$

This is a ring with identity $(0,1)$.

Application: abelianization

Consider the inclusion functor

$$\text{category of abelian groups} \to \textbf{Group}.$$

It has a left adjoint called abelianization, which assigns to every group G the quotient group

$$G^{\mathsf{ab}} = G/[G,G],$$

where $[G,G]$ is the subgroup of G generated by

$$g^{-1}h^{-1}gh, \quad g,h \in G.$$

A functor with both a left and right adjoint

Let **Top** be the category of topological spaces.

Consider the forgetful functor

$$\mathbf{forget}\colon \mathbf{Top} \to \mathbf{Set}.$$

Left adjoint: The functor taking a set to that same set with the discrete topology is left adjoint to **forget**.

Right adjoint: The functor taking a set to that same set with the trivial topology is right adjoint to **forget**.

Conclusion

Adjoint functors are a natural concept in mathematics that lead to many useful constructions:

- free vector spaces
- free (abelian) groups
- abelianization
- discrete/trivial topology

The idea of adjoint functors can be generalized further, to the setting of

- monoidal categories,
- 2-categories.

This is related to higher representation theory.