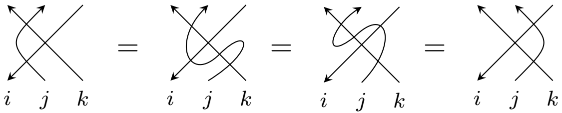
An equivalence between truncations of categorified quantum groups and Heisenberg categories



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Outline

Goals:

1. Categorify the principal realization of the basic representation of $\mathfrak{sl}_\infty$.
2. Relate Heisenberg categories and categorified quantum groups.

Overview:

1. The principal realization
2. Categorification and 2-categories
3. A 2-category $\mathscr{A}$
4. Relation to categorified quantum groups
5. Relation to Heisenberg categories
6. Applications and further directions

Symmetric functions

Let Sym be the $\mathbb{Q}$-algebra of symmetric functions. Recall that

$$\mathrm{Sym} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q} s_\lambda \qquad \text{(as } \mathbb{Q}\text{-vector spaces)},$$

where

- $\mathcal{P}$ is the set of partitions,
- s_λ is the Schur function corresponding to $\lambda \in \mathcal{P}$.

The Heisenberg algebra

The Heisenberg algebra H

Let H be the $\mathbb{Q}$-algebra generated by $p_n^{\pm}$, $n \in \mathbb{N}_+$, with relations

$$p_n^+ p_m^+ = p_m^+ p_n^+, \quad p_n^- p_m^- = p_m^- p_n^-, \quad p_n^- p_m^+ = p_m^+ p_n^- + \delta_{n,m} n.$$

Action of H on Sym

H acts naturally on the algebra Sym of symmetric functions:

- p_n^+ acts by multiplication by power sum p_n,
- p_n^- is adjoint to p_n^+:

$$\langle p_n^- \cdot f, g \rangle = \langle f, p_n g \rangle \quad \forall\, f, g \in \text{Sym}.$$

(We choose the pairing so that the Schur functions are orthonormal.)

This is called the Fock space representation of H.

The algebra $U = U(\mathfrak{sl}_\infty)$

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Let $\mathfrak{sl}_\infty$ be the Lie algebra of

- trace zero infinite matrices $a = (a_{ij})_{i,j\in\mathbb{Z}}$ with rational entries,
- such that the number of nonzero a_{ij} is finite.

Let $U = U(\mathfrak{sl}_\infty)$ be its universal enveloping algebra.

We have the Chevalley generators

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad i \in \mathbb{Z},$$

where $E_{i,j}$ has (i,j)-entry equal to one and all other entries zero.

Partitions

A box of a partition (=Young diagram) in row k and column ℓ has content $\ell - k \in \mathbb{Z}$.

For $\lambda \in \mathcal{P}$, we let

- $\lambda \boxplus i$ be the partition obtained from λ by adding a box of content i if possible (otherwise, set $\lambda \boxplus i = \mathbf{0}$),
- $\lambda \boxminus i$ be the partition obtained from λ by removing a box of content i if possible (otherwise, set $\lambda \boxminus i = \mathbf{0}$).

Example

Let $\lambda = (3, 2) \vdash 5$. Then we have

$$\lambda = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline \text{-1} & 0 & \\ \cline{1-2} \end{array} \quad \lambda \boxplus 3 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline \text{-1} & 0 & & \\ \cline{1-2} \end{array} \quad \lambda \boxminus 0 = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline \text{-1} & & \\ \cline{1-1} \end{array} \quad \lambda \boxplus 5 = \mathbf{0}$$

where the box labels indicate the content.

Basic representation

The basic representation of U corresponds to the action on Sym given by

$$e_i \cdot s_\lambda = s_{\lambda \boxminus i}, \qquad f_i \cdot s_\lambda = s_{\lambda \boxplus i}.$$

(We define $s_{\mathbf{0}} = 0$.)

This yields an irreducible highest weight representation of $\mathfrak{sl}_\infty$ of highest weight Λ_0 (the 0-th fundamental weight).

The principal realization

By the above, we have algebra homomorphisms

$$H \xrightarrow{r_H} \mathrm{End}_{\mathbb{Q}} \mathsf{Sym} \xleftarrow{r_U} U.$$

The images of r_H and r_U are not equal.

However, we have equality of their idempotent modifications:

$$\bigoplus_{\lambda,\mu} 1_\mu r_H(H) 1_\lambda = \bigoplus_{\lambda,\mu} 1_\mu r_U(U) 1_\lambda,$$

where 1_λ is projection onto $\mathbb{Q}s_\lambda \subseteq \mathsf{Sym}$.

This is an $\mathfrak{sl}_\infty$ analogue of fact that the basic representation of $\widehat{\mathfrak{sl}}_n$ remains irreducible when restricted to the principal Heisenberg subalgebra.

This is a crucial ingredient in the principal realization of the basic representation.

Categorification via split Grothendieck group

Suppose $\mathcal{C}$ is an additive category (i.e. have $\oplus$).

$\mathrm{Iso}_{\mathbb{Z}}(\mathcal{C})$ = free abelian group generated by isom. classes of objects in $\mathcal{C}$.

The split Grothendieck group of $\mathcal{C}$ is

$$K(\mathcal{C}) = \mathrm{Iso}_{\mathbb{Z}}(\mathcal{C}) / \langle [X \oplus Y] = [X] + [Y] \mid X, Y \in \mathcal{C} \rangle.$$

If $\mathcal{C}$ is monoidal (i.e. have $\otimes$), then $K(\mathcal{C})$ is a ring:

$$[X] \cdot [Y] = [X \otimes Y].$$

Categorification

For our purposes, to categorify a ring R is to find an additive monoidal category $\mathcal{C}$ such that

$$K(\mathcal{C}) \cong R \quad \text{as rings.}$$

Rings with idempotents as categories

The ring

$$\bigoplus_{\lambda,\mu} 1_\mu r_H(H) 1_\lambda = \bigoplus_{\lambda,\mu} 1_\mu r_U(U) 1_\lambda$$

can naturally be thought of as a category.

Objects: partitions λ.

Morphisms: For partitions λ, μ,

$$\mathrm{Mor}(\lambda, \mu) = 1_\mu r_H(H) 1_\lambda = 1_\mu r_U(U) 1_\lambda.$$

Question: What is the categorification of a category?

Answer: A 2-category!

2-categories

A 2-category has:

- objects,
- 1-morphisms between objects:

$$A \xrightarrow{f} B$$

- 2-morphisms between parallel 1-morphisms:

$$A \underset{g}{\overset{f}{\Downarrow \alpha}} B$$

We have horizontal and vertical composition of 2-morphisms. These are required to satisfy some natural axioms (associativity, etc.).

In particular, for objects A and B, we have a category $\mathrm{Mor}(A, B)$:

- objects of $\mathrm{Mor}(A, B)$ are 1-morphisms from A to B,
- morphisms of $\mathrm{Mor}(A, B)$ are 2-morphisms in the 2-category.

2-categories

Example (Bimodules over rings)

- Objects: Rings
- 1-Morphisms: For rings R, S, the 1-morphisms from R to S are (S, R)-bimodules
- 2-Morphisms: The 2-morphisms are bimodule homomorphisms.

(Split) Grothendieck group of a 2-category

The Grothendieck group of a 2-category $\mathscr{C}$ is a category $K(\mathscr{C})$ with:

- $\operatorname{Ob} K(\mathscr{C}) = \operatorname{Ob} \mathscr{C}$,
- $\operatorname{Mor}_{K(\mathscr{C})}(A, B) = K(\operatorname{Mor}_{\mathscr{C}}(A, B))$.

Our goal

Find 2-categories categorifying the categories appearing in the principal realization.

The 2-category $\mathscr{A}$

We define an additive linear 2-category $\mathscr{A}$ as follows.

Objects: $\mathrm{Ob}\,\mathscr{A} = \mathbb{N}[\mathcal{P}]$, the free monoid on set of partitions $\mathcal{P}$.

We denote the zero object by $\mathbf{0}$.

1-Morphisms: Generated by (i.e. direct sums of compositions of)

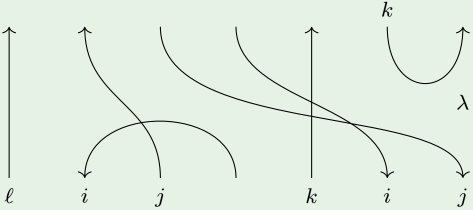
$$\mathsf{F}_i 1_\lambda = 1_{\lambda \boxplus i} \mathsf{F}_i = 1_{\lambda \boxplus i} \mathsf{F}_i 1_\lambda \colon \lambda \to \lambda \boxplus i,$$
$$\mathsf{E}_i 1_\lambda = 1_{\lambda \boxminus i} \mathsf{E}_i = 1_{\lambda \boxminus i} \mathsf{E}_i 1_\lambda \colon \lambda \to \lambda \boxminus i$$

1_λ is the identity morphism on λ.

2-Morphisms: $\mathbb{Q}$-algebra generated by planar string diagrams up to isotopy and modulo local relations. Strands are labeled by integers and regions by partitions.

The 2-category $\mathscr{A}$

Example



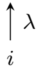
is a 2-morphism

$$\mathsf{F}_\ell \mathsf{E}_i \mathsf{F}_j \mathsf{F}_i \mathsf{F}_k \mathsf{E}_i \mathsf{E}_j 1_\lambda \to \mathsf{F}_\ell \mathsf{F}_j \mathsf{E}_j \mathsf{E}_i \mathsf{F}_k \mathsf{E}_k \mathsf{F}_k 1_\lambda.$$

Horizontal composition of 2-morphisms: horizontal juxtaposition

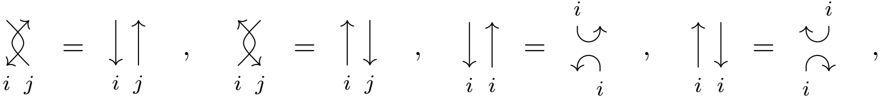
Vertical composition of 2-morphisms: vertical stacking of diagrams

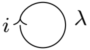
Identity 2-morphisms of $\mathsf{F}_i 1_\lambda$ and $\mathsf{E}_i 1_\lambda$ are:

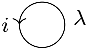


and

The local relations are: For $|i-j|, |i-k|, |j-k| > 1$,





$= \mathrm{id}_\lambda$ if $\lambda \boxminus i \neq \mathbf{0}$,


$= \mathrm{id}_\lambda$ if $\lambda \boxplus i \neq \mathbf{0}$

Truncated categorified quantum groups

Categorified quantum group $\mathscr{U}$ (Cautis, Khovanov, Lauda, Rouquier)

The categorified quantum group $\mathscr{U}$ of type A_∞ is a graded additive linear 2-category.

- Objects: Free monoid generated by weight lattice X of $\mathfrak{sl}_\infty$
- 1-morphisms: Generated by 1_x, $\mathcal{E}_i 1_x$, $\mathcal{F}_i 1_x$, $i \in I$, $x \in X$
- 2-morphisms: Certain dotted planar string diagrams modulo isotopy and local relations

Truncated version $\mathscr{U}^{\text{tr}}$

Idea: We obtain $\mathscr{U}^{\text{tr}}$ from $\mathscr{U}$ by "killing" weights not appearing in the basic representation.

Precise definition: $\mathscr{U}^{\text{tr}}$ is the quotient of $\mathscr{U}$ by the identity 2-morphisms of 1_x for x not a weight of the basic representation.

Connection between $\mathscr{A}$ and $\mathscr{U}$

Proposition

1. The 2-morphism spaces of $\mathscr{U}^{\mathrm{tr}}$ are nonnegatively graded.
2. The positive degree 2-morphism spaces are spanned by diagrams with dots.

Let $\mathscr{U}_0$ be the degree zero piece of $\mathscr{U}^{\mathrm{tr}}$ (i.e. we consider the degree zero piece of the 2-morphism spaces).

Theorem (Queffelec–S.-Yacobi 2017)

The 2-categories $\mathscr{A}$ and $\mathscr{U}_0$ are equivalent via a 2-functor that acts on 2-morphisms (planar string diagrams) by reversing the orientation of strands.

Summary: We can view the 2-category $\mathscr{A}$ as the degree zero piece of a truncation of the categorified quantum group $\mathscr{U}$.

The truncated Heisenberg 2-category

Defined an additive linear 2-category $\mathscr{H}^{\mathrm{tr}\prime}$ as follows.

Objects: $\mathrm{Ob}\,\mathscr{A} = \mathbb{N}[\mathbb{N}]$, the free monoid on $\mathbb{N}$.

1-Morphisms: Generated by (i.e. direct sums of compositions of)

$$\mathsf{Q}_+ 1_k \colon k \to k+1,$$
$$\mathsf{Q}_- 1_k \colon k \to k-1.$$

1_k is the identity morphism on k.

2-Morphisms: $\mathbb{Q}$-algebra generated by planar string diagrams up to isotopy and modulo local relations. Strands are not labeled, while regions are labeled by elements of $\mathbb{N}$.

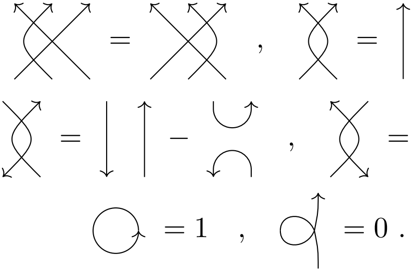
Note: $\mathscr{H}^{\mathrm{tr}\prime}$ is a truncation of a larger category $\mathscr{H}'$ with objects $\mathbb{N}[\mathbb{Z}]$.

The Heisenberg category $\mathscr{H}^{\mathrm{tr}\prime}$: planar diagrammatics

Identity 2-morphisms of $\mathsf{Q}_+ 1_k$ and $\mathsf{Q}_- 1_k$ are:

$$\uparrow k \qquad \text{and} \qquad \downarrow k$$

The local relations are:



Note: These relations come from the behaviour of induction and restriction of representations for symmetric groups:

$$\mathrm{Res} \circ \mathrm{Ind} \cong \mathrm{Ind} \circ \mathrm{Res} \oplus \mathrm{id}\,.$$

Idempotent completion (categories)

Definition: idempotent completion (Karoubi envelope)

The idempotent completion of a category $\mathcal{C}$ is the category whose

- objects are pairs (A, e) where $A \in \operatorname{Ob} \mathcal{C}$ and $e \in \operatorname{Mor} \mathcal{C}$ is an idempotent ($e^2 = e$), and
- morphisms from (A, e) to (B, f) are $\psi \in \operatorname{Mor}_{\mathcal{C}}(A, B)$ such that

$$f\psi = \psi = \psi e.$$

Intuition: One thinks of passing to the idempotent completion as adding in objects such that the idempotents correspond to projections onto direct summands.

$$A \cong X \oplus Y \twoheadrightarrow X \hookrightarrow X \oplus Y \cong A$$

$$e : X \oplus Y \to X \oplus Y$$

Example: If R is a ring, the idempotent completion of the category of free R-modules is the category of projective R-modules.

Idempotent completion (2-categories)

Now suppose $\mathscr{C}$ is a 2-category.

Then, for objects A and B, $\mathrm{Mor}_{\mathscr{C}}(A, B)$ is a category.

Usual idempotent completion of 2-categories

The idempotent completion of a 2-category is usually defined in the categorification literature as follows:

- The objects of the completion are the objects of $\mathscr{C}$.
- The morphism categories of the completion are the idempotent completions of the morphism categories of $\mathscr{C}$.

Idea: One leaves objects alone and takes the usual idempotent completion of morphism categories.

A larger idempotent completion

For a 2-category $\mathscr{C}$, we define an idempotent completion $\mathrm{Kar}(\mathscr{C})$:

- Objects: triples $(x, \mathsf{e}, \epsilon)$, where
  - $x \in \mathrm{Ob}\,\mathscr{C}$,
  - $\mathsf{e} \colon x \to x$ is an idempotent 1-morphism in $\mathscr{C}$, and
  - $\epsilon \colon \mathsf{e} \to \mathsf{e}$ is an idempotent 2-morphism (under vertical comp) in $\mathscr{C}$.
- 1-Morphisms: $(\mathsf{g}, \beta) \colon (x, \mathsf{e}, \epsilon) \to (x', \mathsf{e}', \epsilon')$ with
  - $\mathsf{g} \colon x \to x'$ a 1-morphism in $\mathscr{C}$ such that $\mathsf{e}'\mathsf{g}\mathsf{e} = \mathsf{g}$ and
  - $\beta \colon \mathsf{g} \to \mathsf{g}$ an idempotent 2-morphism in $\mathscr{C}$ such that $\epsilon'\beta\epsilon = \beta$.
- 2-Morphisms: The 2-morphisms between parallel 1-morphisms

$$(\mathsf{g}, \beta), (\mathsf{h}, \gamma) \colon (x, \mathsf{e}, \epsilon) \to (x', \mathsf{e}', \epsilon')$$

  are 2-morphisms $\alpha \colon \mathsf{g} \to \mathsf{h}$ in $\mathscr{C}$ such that $\gamma \circ \alpha \circ \beta = \alpha$.

Let $\mathscr{H}^{\mathrm{tr}} = \mathrm{Kar}(\mathscr{H}^{\mathrm{tr}\prime})$.

Note: This idempotent completion is different (larger) than the one usually considered in the categorification literature. In general, it has more objects.

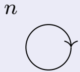
Connection between $\mathscr{A}$ and $\mathscr{H}^{\mathrm{tr}}$

Proposition (Queffelec–S.–Yacobi 2017)

We have an equivalence of 2-categories

$$\mathscr{H}^{\mathrm{tr}} \cong \mathscr{H}_\epsilon \oplus \mathscr{H}_\delta,$$

where $\mathscr{H}_\epsilon$ is the category obtained from $\mathscr{H}^{\mathrm{tr}}$ by imposing one additional local relation:



$$= n, \quad n \in \mathbb{N}.$$

Theorem (Queffelec–S.–Yacobi 2017)

We have an equivalence of 2-categories

$$\mathscr{A} \cong \mathscr{H}_\epsilon.$$

Note: The equivalence relies crucially on the larger idempotent completion.

Big picture

We have 2-functors

$$\mathscr{H} \xrightarrow[\text{complete}]{\text{truncate}} \mathscr{H}^{\text{tr}} \xrightarrow{\text{summand}} \mathscr{H}_\epsilon \cong \mathscr{A} \cong \mathscr{U}_0 \xleftarrow{\text{summand}} \mathscr{U}^{\text{tr}} \xleftarrow{\text{truncate}} \mathscr{U}\,.$$

These categorify the algebra homomorphisms mentioned earlier:

$$H \xrightarrow{r_H} \mathrm{End}_{\mathbb{Q}} \mathsf{Sym} \xleftarrow{r_U} U.$$

In particular:

- $\mathscr{H}$ (conjecturally) categorifies H,
- $\mathscr{U}$ categorifies U,
- $\mathscr{A}$ categorifies $\mathrm{End}_{\mathbb{Q}} \mathsf{Sym}$,
- The isomorphisms $\mathscr{H}_\epsilon \cong \mathscr{A} \cong \mathscr{U}_0$ are a categorical analogue of the principal realization of the basic representation of $\mathfrak{sl}_\infty$.

Induced actions

Our results imply the following:

Categorical restriction of basic rep to principal Heisenberg algebra

Any additive linear 2-functor $\mathscr{U} \to \mathscr{C}$ categorifying the basic representation induces a categorical action $\mathscr{H} \to \mathscr{C}$.

Categorical induced action

Any additive linear 2-functor $\mathscr{H} \to \mathscr{C}$ mapping all objects $n \in \mathbb{Z}$, $n < 0$, to zero induces a categorical action $\mathscr{U} \to \mathscr{C}$.

Example

- The category $\mathscr{H}$ acts naturally on modules for symmetric groups.
- This induces an action of the categorified quantum group $\mathscr{U}$ on modules for symmetric groups.
- This relates the above constructions to work of Brundan and Kleshchev.

Further directions

Positive characteristic

- Should be able to replace $\mathbb{Q}$ by a field of characteristic $p > 0$.
- Should correspond to replacing $\mathfrak{sl}_\infty$ by $\widehat{\mathfrak{sl}}_p$.

Cyclotomic Hecke algebras

- Group algebras of symmetric groups are level one degenerate cyclotomic Hecke algebras.
- Generalize to arbitrary level.
- Would involve higher level Heisenberg categories (Mackaay-S.).

More general Heisenberg categories

- One can associated a Heisenberg category to any graded Frobenius superalgebra (Rosso-S.).
- Results should generalize to this setting.