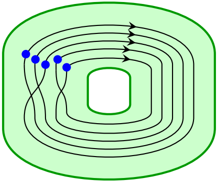
# A graphical calculus for the Jack inner product on symmetric functions



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Slides available online: alistairsavage.ca/talks

Paper: arXiv:1610.01862 (to appear in *J. Comb. Theory A*)

# Outline

Goal: Develop a graphical realization of the Jack inner product on symmetric functions.

Overview:

1. Review of symmetric functions & bilinear forms
2. Heisenberg algebras
3. Heisenberg categories
4. Planar diagrammatics: closed and annular diagrams
5. Diagrammatic bilinear form
6. Further directions

# Power-sum and monomial symmetric functions

Let Sym be the $\mathbb{C}$-algebra of symmetric functions.

For $n \geq 1$, let

$$p_n = x_1^n + x_2^n + x_3^n + \cdots$$

be the $n$-th power sum.

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$, define the power-sum symmetric function

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

We also define the monomial symmetric function

$$m_\lambda = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots ,$$

where the sum is over all permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of $\lambda$.

# Jack inner products

Fact: The $p_\lambda$ form a basis of Sym.

Result: We can define an inner product on Sym by specifying its values on the $p_\lambda$.

## Jack inner product

Fix $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Define the Jack inner product by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} z_\lambda,$$

where $\ell(\lambda)$ is the length of $\lambda$ and

$$z_\lambda = \prod_{k \geq 1} k^{m_k(\lambda)} m_k(\lambda)!,$$

where $m_k(\lambda)$ is the number of parts of $\lambda$ equal to $k$.

Aside: Form is uniquely determined by its values on the $p_n$ and the fact that it is a Hopf pairing.

# Jack symmetric functions

Fact: The $m_\lambda$ form a basis of Sym.

Dominance order: For partitions $\lambda, \mu$ we say

$$\mu \le \lambda \iff \mu_1 + \cdots + \mu_i \le \lambda_1 + \cdots + \lambda_i \text{ for all } i \ge 1.$$

## Jack symmetric functions

The Jack symmetric functions $J_\lambda$ are uniquely defined by the conditions

1. $J_\lambda \in m_\lambda + \mathrm{Span}\{m_\mu \mid \mu < \lambda\}$,
2. $\langle J_\lambda, J_\mu \rangle_\alpha = 0$ if $\lambda \neq \mu$.

Aside: The Jack functions are obtained from the Macdonald symmetric functions via the specialization

$$q = t^\alpha, \qquad t \to 1.$$

# The Heisenberg algebra

For the remainder of the talk, we fix $\alpha \in \mathbb{Z}$, $\alpha \neq 0$.

**The Heisenberg algebra $\mathfrak{h}_\alpha$**

Let $\mathfrak{h}_\alpha$ be the $\mathbb{C}$-algebra generated by $p_n^\pm$, $n \in \mathbb{N}_+$, with relations

$$p_n^+ p_m^+ = p_m^+ p_n^+, \quad p_n^- p_m^- = p_m^- p_n^-, \quad p_n^- p_m^+ = p_m^+ p_n^- + \delta_{n,m} n\alpha.$$

As a vector space, we have

$$\mathfrak{h}_\alpha = \mathfrak{h}_\alpha^+ \otimes \mathfrak{h}_\alpha^-,$$

where

$$\mathfrak{h}_\alpha^\pm = \langle p_n^\pm \mid n \geq 1 \rangle \cong \mathsf{Sym}$$

are subalgebras.

# Action of the Heisenberg algebra on Sym

## Action of $\mathfrak{h}_\alpha$ on Sym

$\mathfrak{h}_\alpha$ acts naturally on Sym:

- $p_n^+$ acts by multiplication by $p_n$,
- $p_n^-$ is adjoint to $p_n^+$:

$$\langle p_n^- \cdot f, g \rangle_\alpha = \langle f, p_n g \rangle_\alpha \quad \forall\, f, g \in \mathsf{Sym}.$$

This is called the Fock space representation of $\mathfrak{h}_\alpha$.

# Frobenius algebras

## Frobenius algebras

A graded Frobenius algebra is a f.d. $\mathbb{N}$-graded algebra

$$B = \bigoplus_{n \in \mathbb{N}} B_n,$$

with a trace map $\operatorname{tr}\colon B \to \mathbb{C}$ satisfying:

- $\operatorname{tr}$ is linear,
- the kernel of $\operatorname{tr}$ contains no nonzero left ideal.

## Example

Let $B = \mathbb{C}[y]/(y^k)$, graded by degree.

Then $B$ is a Frobenius algebra with trace map

$$\operatorname{tr}\left(a_0 + a_1 y + \cdots + a_{k-1} y^{k-1}\right) = a_{k-1}.$$

# The Heisenberg algebra $\mathfrak{h}_B$

For the rest of the talk, fix a graded Frobenius algebra

$$B = \bigoplus_{n=1}^{\delta} B_n, \quad B_\delta \neq 0.$$

Its graded dimension is

$$\operatorname{grdim} B = \sum_{n \in \mathbb{N}} q^n \dim B_n \in \mathbb{N}[q].$$

Simplifying assumption (for this talk): $B_0 = \mathbb{C}$, $\delta > 0$.

## The Heisenberg algebra $\mathfrak{h}_B$

Let $\mathfrak{h}_B$ be the $\mathbb{C}(q)$-algebra generated by $p_n^{\pm}$, $n \in \mathbb{N}_+$, with relations

$$p_n^+ p_m^+ = p_m^+ p_n^+, \quad p_n^- p_m^- = p_m^- p_n^-, \quad p_n^- p_m^+ = p_m^+ p_n^- + \delta_{n,m} n \operatorname{grdim} B.$$

Note: $\mathfrak{h}_B|_{q=1} = \mathfrak{h}_\alpha$, with $\alpha = \dim B$.

# The Heisenberg category

We define a $\mathbb{C}$-linear strict monoidal category $\mathcal{H}_B$ as follows:

The objects of $\mathcal{H}_B$ are generated by two objects $\mathsf{Q}_+$ and $\mathsf{Q}_-$, together with their degree shifts.

So objects are sequences of $\mathsf{Q}_+$ and $\mathsf{Q}_-$, which a $\mathbb{Z}$-degree shift:

$$\varnothing, \quad \{2\}\mathsf{Q}_+, \quad \{-4\}\mathsf{Q}_-\mathsf{Q}_-\mathsf{Q}_+\mathsf{Q}_+\mathsf{Q}_+\mathsf{Q}_-, \quad \text{etc.}$$

The morphisms are $\mathbb{C}$-linear combinations of planar diagrams:

- oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0,1]$,
- agree with the domain/codomain at the top/bottom of the diagram,
- up to isotopy, and
- modulo certain local relations.

# Morphisms: planar diagrams

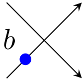
We have the identity morphisms:

$$\mathrm{id}_{\mathsf{Q}_+} = \uparrow \qquad\qquad \mathrm{id}_{\mathsf{Q}_-} = \downarrow$$

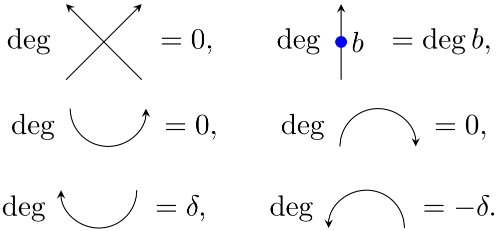
Strands are allowed to carry dots labeled by elements of $B$:


Collision of dots is given by multiplication in $B$:


Strands are allowed to cross:

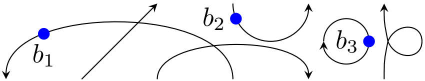


The morphism spaces are graded:




(Recall $\delta$ is the top degree of $B$.)

The degree of a morphism determines the relative degree shifts of the domain and codomain.

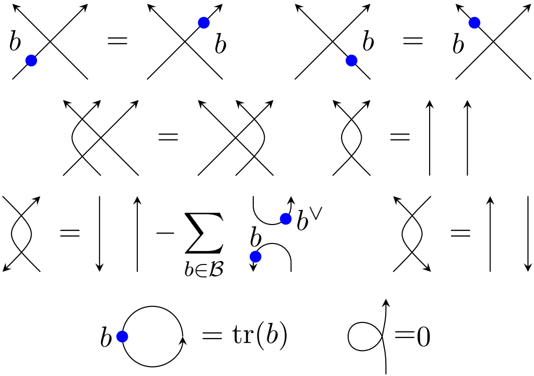



is a morphism

$$\{\deg b_1 + \deg b_2 + \deg b_3 + \delta\}\mathsf{Q}_-\mathsf{Q}_+\mathsf{Q}_+\mathsf{Q}_+\mathsf{Q}_-\mathsf{Q}_+ \to \mathsf{Q}_+\mathsf{Q}_-\mathsf{Q}_+\mathsf{Q}_+$$

# Morphisms: local relations

The local relations we impose are:




Here $\mathcal{B}$ is a basis of $B$ and $\{b^\vee \mid b \in \mathcal{B}\}$ is the (right) dual basis:

$$\mathrm{tr}(bc^\vee) = \delta_{b,c}.$$

# Remark: Grothendieck group categorification

Recall: The Karoubi envelope (or idempotent completion) is an "enlargement" of a category $\mathcal{C}$, with one object for each idempotent morphism in $\mathcal{C}$.

Let $\operatorname{Kar}\mathcal{H}_B$ be the Karoubi envelope of the graphical category $\mathcal{H}_B$.

**Theorem (Rosso–S. 2015)**

There is a natural algebra isomorphism

$$\mathfrak{h}_B \to \text{split Grothendieck group of } \operatorname{Kar}\mathcal{H}_B.$$

Hence $\operatorname{Kar}\mathcal{H}_B$ categorifies $\mathfrak{h}_B$.

# Natural action of $\mathcal{H}_B$

For $n \geq 1$, consider the wreath product algebra

$$B^{\otimes n} \rtimes S_n.$$

As a vector space, $B^{\otimes n} \rtimes S_n = B^{\otimes n} \otimes \mathbb{C}S_n$.

The factors $B^{\otimes n}$ and $\mathbb{C}S_n$ are subalgebras, and

$$wb = (w \cdot b)w, \qquad b \in B^{\otimes n},\ w \in S_n,$$

where $w \cdot b$ is action of $w$ on $b$ by permuting factors.

### Action of $\mathcal{H}_B$ on $\bigoplus_n B^{\otimes n} \rtimes S_n$-mod

- $\mathsf{Q}_+$ acts by induction $B^{\otimes n} \rtimes S_n\text{-mod} \to B^{\otimes (n+1)} \rtimes S_{n+1}$,
- $\mathsf{Q}_+$ acts by restriction $B^{\otimes (n+1)} \rtimes S_{n+1} \to B^{\otimes n} \rtimes S_n\text{-mod}$,
- planar diagrams are natural transformations between compositions of induction and restriction.

# Center of a monoidal category

Suppose $\mathcal{C}$ is an additive monoidal category.

Then $\mathcal{C}$ has an identity object $\mathbf{1}$:

$$X \otimes \mathbf{1} \cong X \cong \mathbf{1} \otimes X \quad \text{for all objects } X \text{ in } \mathcal{C}.$$

## Center of a monoidal category

The center of $\mathcal{C}$ is

$$Z(\mathcal{C}) := \mathrm{End}_{\mathcal{C}}(\mathbf{1}).$$

The center is naturally a ring, under sum and tensor product of morphisms.

## Example

If $R$ is a ring and $\mathcal{C}$ is the category of $(R, R)$-bimodules, then

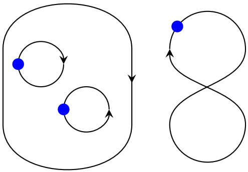
$$Z(\mathcal{C}) = \mathrm{End}_{\mathcal{C}}(R) \cong Z(R).$$

# Closed diagrams

The identity object of $\mathcal{H}_B$ is $\varnothing$.

So the center $Z(\mathcal{H}_B)$ consists of linear combinations of closed diagrams.

Multiplication is given by juxtaposition.



This algebra is clearly commutative (since we consider diagrams up to isotopy).

# Trace of a monoidal category

Suppose $\mathcal{C}$ is a $\mathbb{C}$-linear monoidal category.

The trace (or zeroth Hochschild homology) is

$$\operatorname{Tr}(\mathcal{C}) = \left( \bigoplus_{x \in \operatorname{Ob} \mathcal{C}} \operatorname{End}_{\mathcal{C}}(x) \right) / \operatorname{Span}\{fg - gf \mid f \colon x \to y,\ g \colon y \to x\}.$$

**Example**

Suppose $A$ is an algebra. Let $\mathcal{C}$ be the monoidal category with

- one object $\star$,
- $\operatorname{Hom}_{\mathcal{C}}(\star, \star) = A$,
- tensor product of morphisms given by the product in $A$.

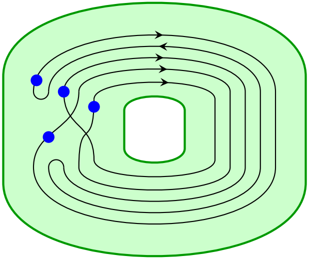
Then

$$\operatorname{Tr}(\mathcal{C}) = A / \operatorname{Span}\{ab - ba \mid a, b \in A\}$$

is the trace (or cocenter) of $A$.

# Annular diagrams

The trace $\mathrm{Tr}(\mathcal{H}_B)$ of $\mathcal{H}_B$ can be identified the space of annular diagrams.

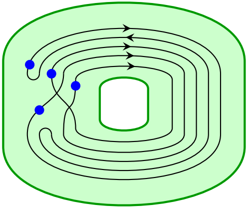


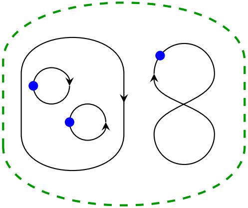
This is an algebra, with product given by nesting.

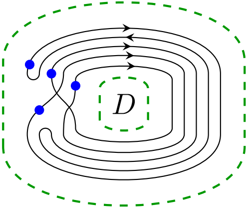
If $A_1$ and $A_2$ are annular diagrams, then $A_1 A_2$ is the annular diagram obtained by placing $A_2$ in the center region of $A_1$.

# Diagrammatic action

The algebra $\mathrm{Tr}(\mathcal{H}_B)$ of annular diagrams acts on the space $Z(\mathcal{H}_B)$ of closed diagrams.

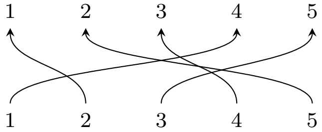
$A =$ 

$D =$ 

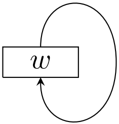
$A \cdot D =$ 

# Closures of permutations

Identify permutations with diagrams:



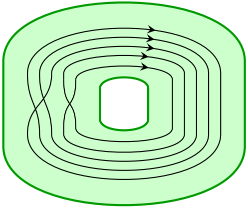
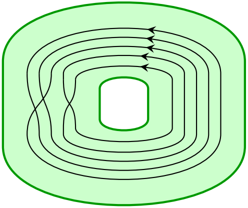

Given a permutation $w$, we can close off to the right to get a closed diagram or an annular diagram:



# Key elements in the trace

More generally, any element of $\mathbb{C}S_n$ can be drawn as a linear combination of braid-like diagrams, on upward or downward strands.

Closing off to the right yields an annular diagram.

$P^+_{(3,2)} =$  $\quad P^-_{(3,2)} =$ 

So to any $f \in \mathbb{C}S_n$, we have the associated annular diagram $[f^\pm]$. Let

$$P^\pm_\lambda = [w^\pm_\lambda],$$

where $w_\lambda$ is a permutation of cycle type $\lambda$.

Question: Why is this well defined?

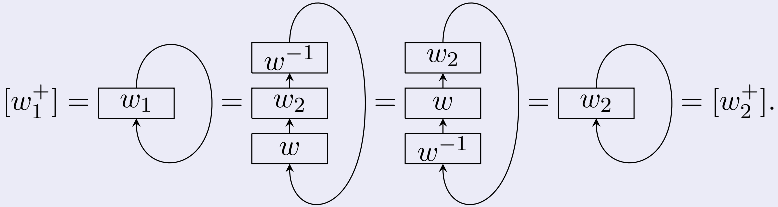
# Example of a diagrammatic proof

## Lemma

If $w_1, w_2 \in S_n$ are conjugate, then $[w_1^{\pm}] = [w_2^{\pm}]$.

## Proof.

Suppose $w_1 = w w_2 w^{-1}$. Then




□

# Identification of the trace

**Theorem (Licata–Rosso–S. 2016)**

We have an isomorphism of algebras

$$\mathrm{Tr}(\mathcal{H}_B) \cong \mathfrak{h}_B, \qquad P_\lambda^\pm \mapsto p_\lambda^\pm.$$

Note: We also have

$$[e_\lambda^\pm] \mapsto s_\lambda^\pm,$$

where $e_\lambda \in \mathbb{C}S_n$ is the primitive idempotent corresponding to a partition $\lambda$ and $s_\lambda$ is the Schur function.

# The diagrammatic bilinear form

Let $\mathrm{Tr}(\mathcal{H}_B)^{\pm}$ be the algebra of annular diagrams spanned by closing off endomorhpisms of $(\mathsf{Q}_{\pm})^n$ for $n \in \mathbb{N}$.

So

$$\mathrm{Tr}(\mathcal{H}_B)^{+} = \text{ algebra of clockwise annular diagrams,}$$
$$\mathrm{Tr}(\mathcal{H}_B)^{-} = \text{ algebra of counterclockwise annular diagrams.}$$

Under the isomorphism $\mathrm{Tr}(\mathcal{H}_B) \cong \mathfrak{h}_B$, we have

$$\mathrm{Tr}(\mathcal{H}_B)^{\pm} \mapsto \mathfrak{h}_B^{\pm} := \langle p_n^{\pm} \mid n \in \mathbb{N} \rangle \cong \mathsf{Sym}.$$

So a pairing

$$\mathrm{Tr}(\mathcal{H}_B)^{-} \times \mathrm{Tr}(\mathcal{H}_B)^{+} \to \mathbb{C}$$

corresponds to a bilinear form on Sym.

# The diagrammatic bilinear form

Fact 1: The space $Z(\mathcal{H}_B)$ of closed diagrams is nonnegatively graded.

Fact 2: The only degree zero closed diagram is the empty diagram.

So we have the projection onto degree 0:

$$\mathbf{F}_0 \colon Z(\mathcal{H}_B) \to \mathbb{C}.$$

## The diagrammatic pairing

We define a pairing

$$\langle -, - \rangle_B \colon \mathrm{Tr}(\mathcal{H}_B)^- \times \mathrm{Tr}(\mathcal{H}_B)^+ \to \mathbb{C},$$
$$\langle x, y \rangle_B = \mathbf{F}_0\big((xy) \cdot 1_\varnothing\big),$$

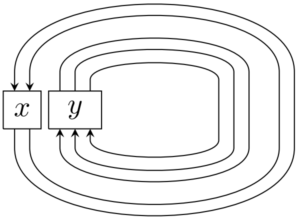
where $1_\varnothing$ is the empty diagram.

# The diagrammatic pairing

Graphically, $\langle x, y\rangle$ is obtained by

- placing the annular diagram $y$ inside the annular diagram $x$,
- viewing the resulting annular diagram as a closed diagram,
- projecting onto degree zero.

$$\langle x, y\rangle_B = \mathbf{F}_0 \left( \quad \right) .$$




Note: If $f, g \in \mathbb{C}S_n$, then the nested diagram $[f][g] \cdot 1_\varnothing$ is already in degree zero and $\mathbf{F}_0$ is unnecessary.

# The diagrammatic pairing

**Theorem (Licata–Rosso–S.)**

Under the isomorphisms $\mathrm{Tr}(\mathcal{H}_B)^{\pm} \cong \mathrm{Sym}$, the diagrammatic form corresponds to the Jack bilinear form at parameter

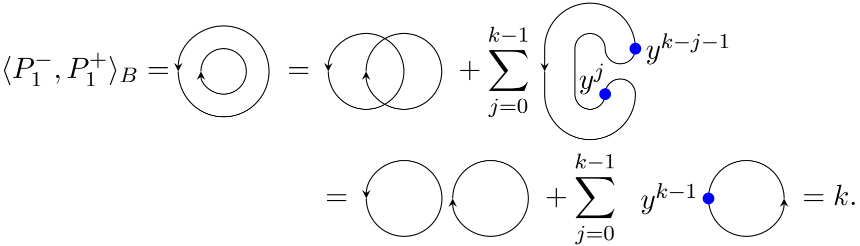
$$\alpha = \dim B.$$

So we have a categorification of the Jack bilinear form, with the Jack parameter categorified by the graded Frobenius algebra $B$.

# Example

Suppose $B = \mathbb{C}[y]/(y^k)$. Trace map is the coefficient of $y^{k-1}$.

The first power sum $p_1$ corresponds to the clockwise circle $P_1^+$ and counterclockwise circle $P_1^-$.




This agrees with the inner product of $p_1$ with itself in the Jack inner product at Jack parameter $k = \dim B$.

# Full generality

The full results are more general than presented in this talk.

### The Frobenius algebra $B$

- In general, $B$ can be a graded Frobenius superalgebra.
- We don't need $B_0 = \mathbb{C}$.
- We only assume that all simple $B$-modules are of type $M$ (i.e. not isomorphic to their parity shifts) and that the trace map is supersymmetric and even.
- The Jack parameter $\alpha$ corresponds to $\dim B_{\text{even}} - \dim B_{\text{odd}}$.

# Further directions I

## More general Frobenius algebras

- The categories $\mathcal{H}_B$ are defined for an arbitrary graded Frobenius superalgebra $B$.
- Allowing $B$ to have simple modules of type $Q$ (isomorphic to their own parity shifts) would result in the space of Schur $Q$-functions.
- Allowing the trace map to not be supersymmetric would introduce twisted Heisenberg algebras.

## Connections to W-algebras

- $\mathrm{Tr}(\mathcal{H}_{\mathbb{C}})$ is isomorphic to a quotient of the W-algebra $W_{1+\infty}$ (Cautis–Licata–Lauda–Sussan)
- $\mathrm{Tr}(\mathcal{H}_B)$ should be related to W-algebras associated to the lattice $K_0(B\text{-mod})$.

# Further directions II

## Wreath product algebras

- $\mathcal{H}_B$ acts on modules for wreath product algebras $B^{\otimes n} \rtimes S_n$.
- Thus, the Heisenberg algebra $\mathrm{Tr}(\mathcal{H}_B) \cong \mathfrak{h}_B$ acts on the centers of these module categories.
- Can therefore use diagrammatics to study the centers of these categories.

## Jack symmetric functions

- We have categorified the Jack inner product.
- Question: Can we categorify the Jack symmetric functions themselves?
- Find natural annular diagrams that correspond to these functions.

# Further directions III

## Geometry: Hilbert schemes

- Equivariant $K$-theory of the Hilbert scheme of points on $\mathbb{C}^2$ is related to the Macdonald ring of symmetric functions (Haiman).
- Equivariant homology related to Jack symmetric functions (Nakajima, Li–Qin–Wang).
- "$K$-theory versus homology" is analogous to "Grothendieck group versus trace".
- So current work should be related to these geometric constructions.