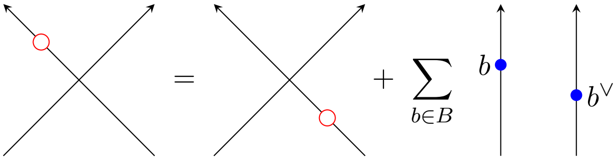
Affine wreath product algebras



Alistair Savage
University of Ottawa

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Outline

Goals: Unify and generalize the theory of algebras defined by "degenerate affine Hecke algebra" type relations.

Overview:

1. Background and motivation
2. Frobenius algebras
3. Affine wreath product algebras
4. Structure theory
5. Representation theory
6. Cyclotomic quotients
7. Further directions

Background and motivation

Degenerate affine Hecke algebras

Fix a commutative ring $\Bbbk$.

The degenerate affine Hecke algebra $\mathcal{H}_n$ of type A is

$$\Bbbk[x_1, \dots, x_n] \otimes \Bbbk S_n$$

as a $\Bbbk$-module.

The factors $\Bbbk[x_1, \dots, x_n]$ and $\Bbbk S_n$ are subalgebras, and

$$\begin{aligned} s_i x_j &= x_j s_i, && j \neq i, i+1, \\ s_i x_i &= x_{i+1} s_i - 1, \end{aligned}$$

where $s_i = (i, i+1)$ is the simple transposition.

$\mathcal{H}_n$ is a "degeneration" $(q \to 1)$ of the affine Hecke algebra.

Degenerate affine Hecke algebras

Surprisingly, modular branching rules for $\mathcal{H}_n$ and their cyclotomic quotients are related to affine Lie algebras of type A (Ariki, Grojnowski, Vazirani, Mathas, Lascoux, Leclerc, Thibon, ...).

Brundan and Kleshchev have related cyclotomic quotients to quiver Hecke algebras (Khovanov–Lauda–Rouquier algebras).

Algebras related to $\mathcal{H}_n$ have appeared in other places:

- affine Sergeev algebra (degenerate affine Hecke–Clifford algebra),
- wreath Hecke algebra,
- affine zigzag algebra.

It would be nice to unify (and generalize) the treatments of these algebras.

Appearances in categorification

The $\mathcal{H}_n$ appeared in endomorphism spaces of categories introduced by Khovanov to categorify the Heisenberg algebra.

Khovanov's categories have been generalized (Rosso–S.). One obtains a Heisenberg category for every $\mathbb{Z}$-graded Frobenius superalgebra.

In the endomorphism spaces of these more general categories, one finds a large family of algebras that specialize to many well-studied analogues of $\mathcal{H}_n$.

Goal

Study the structure and representation theory of this large family of algebras.

Terminology: In this talk, all algebras and modules are super and $\mathbb{Z}$-graded.

Frobenius algebras

Frobenius algebras: Definition

Definition 1 (trace map)

A Frobenius algebra is a f.d. associative algebra F together with a linear trace map

$$\mathrm{tr}\colon F \to \Bbbk$$

such that $\ker \mathrm{tr}$ contains no nonzero left ideals.

Definition 2 (bilinear form)

A Frobenius algebra is a f.d. associative algebra F together with a nondegenerate bilinear form satisfying

$$\langle fg, h \rangle = \langle f, gh \rangle \quad \text{for all } f, g, h \in F.$$

The connection between the two definitions is given by

$$\langle f, g \rangle = \mathrm{tr}(fg).$$

For simplicity, we assume tr is an even map.

Frobenius algebras: Examples

Example ($\Bbbk$)

$\Bbbk$ is a Frobenius algebra with $\mathrm{tr} = \mathrm{id}_{\Bbbk}$.

Example (Clifford algebra)

The Clifford algebra

$$\mathrm{Cl} = \Bbbk[c]/(c^2 - 1), \qquad \bar{c} = 1,$$

is a Frobenius algebra with $\mathrm{tr}(1) = 1$, $\mathrm{tr}(c) = 0$.

Example ($\Bbbk[x]/(x^k)$)

$\Bbbk[x]/(x^k)$ is a Frobenius algebra with

$$\mathrm{tr}(x^\ell) = \delta_{\ell,k-1},$$

We can give it nontrivial $\mathbb{Z}$-grading by setting $|x| = 1$.

Frobenius algebras: Examples

Example (Matrix algebra)

Any matrix algebra over a field is a Frobenius algebra with the usual trace.

Example (Group algebra)

Suppose G is a finite group and fix $h \in G$.

The group algebra $\Bbbk G$ is a Frobenius algebra with

$$\mathrm{tr}(g) = \delta_{g,h}, \quad g \in G.$$

Standard choice: $h = 1_G$.

Example (Hopf algebras)

Every f.d. Hopf algebra is a Frobenius algebra.

From now on: F is a Frobenius algebra with trace tr.

Frobenius algebras: Nakayama automorphism

The Nakayama automorphism is the algebra automorphism $\psi \colon F \to F$ defined by

$$\operatorname{tr}(fg) = (-1)^{\bar{f}\bar{g}} \operatorname{tr}\big(g\psi(f)\big), \quad f, g \in F.$$

We say F is symmetric if $\psi = \mathrm{id}_F$.

Examples

F	tr	ψ
$\Bbbk$	id	id
Cl	$\operatorname{tr}(1) = 1$, $\operatorname{tr}(c) = 0$	$c \mapsto -c$
$\Bbbk[x]/(x^k)$	$\operatorname{tr}(x^\ell) = \delta_{\ell,k-1}$	id
matrix algebra	tr	id
$\Bbbk G$	$\operatorname{tr}(g) = \delta_{g,h}$	$g \mapsto h^{-1}gh$

Assumption: the characteristic of $\Bbbk$ does not divide the order of ψ.

Frobenius algebras: dual bases

Fix a basis B of F. The left dual basis is

$$B^\vee = \{b^\vee \mid b \in B\}$$

defined by

$$\mathrm{tr}\left(b^\vee c\right) = \delta_{b,c}, \qquad b, c \in B.$$

It is easy to check that

$$\sum_{b \in B} b \otimes b^\vee \in F \otimes F$$

is independent of the basis B.

Let δ be the top nonzero degree of F. Then

- tr has degree $-\delta$,
- $\sum_{b \in B} b \otimes b^\vee$ is even of degree δ.

Affine wreath product algebras

Wreath product algebras

The symmetric group S_n acts on $F^{\otimes n}$ by superpermutations:

$$s_i \cdot (f_1 \otimes \cdots \otimes f_n) = (-1)^{\bar{f}_i \bar{f}_{i+1}} f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n,$$

We let ${}^{\pi}\mathbf{f}$ denote the action of $\pi \in S_n$ on $\mathbf{f} \in F^{\otimes n}$.

Wreath product algebra

The wreath product algebra is

$$F^{\otimes n} \rtimes S_n = F^{\otimes n} \otimes \Bbbk S_n$$

as $\Bbbk$-modules. Multiplication is determined by

$$(\mathbf{f}_1 \otimes \pi_1)(\mathbf{f}_2 \otimes \pi_2) = \mathbf{f}_1 \, {}^{\pi_1}\mathbf{f}_2 \otimes \pi_1 \pi_2.$$

Wreath product algebras: Examples

Example ($F = \Bbbk$)

$\Bbbk^{\otimes n} \rtimes S_n \cong \Bbbk S_n$

Example ($F = \mathrm{Cl}$)

$\mathrm{Cl}^{\otimes n} \rtimes S_n$ is the Sergeev algebra, which plays an important role in the projective representation theory of the symmetric group.

Example ($F = \Bbbk G$, $G = \mathbb{Z}/2\mathbb{Z}$)

When G is a cyclic group of order 2, $(\Bbbk G)^{\otimes n} \rtimes S_n$ is the group algebra of the hyperoctahedral group, the Weyl group of type B.

Example ($F = \Bbbk G$, $G = \mathbb{Z}/r\mathbb{Z}$)

When G is a cyclic group of order r, $(\Bbbk G)^{\otimes n} \rtimes S_n$ is the group algebra of the complex reflection group $G(r, 1, n)$.

Affine wreath product algebras: Definition

Fix $n \in \mathbb{N}$. For $f \in F$ and $1 \leq i \leq n$, define

$$f_i = 1^{\otimes(i-1)} \otimes f \otimes 1^{\otimes(n-i)} \in F^{\otimes n},$$
$$\psi_i = \mathrm{id}^{\otimes(i-1)} \otimes \psi \otimes \mathrm{id}^{\otimes(n-i)} \colon F^{\otimes n} \to F^{\otimes n}.$$

We define the affine wreath product algebra $\mathcal{A}_n(F)$ to be the $\mathbb{Z}$-graded superalgebra that is the free product of $\Bbbk$-algebras

$$\Bbbk[x_1, \dots, x_n] \star F^{\otimes n} \star \Bbbk S_n,$$

modulo the relations

$$\begin{aligned}
\mathbf{f} x_i &= x_i \psi_i(\mathbf{f}), && 1 \leq i \leq n,\ \mathbf{f} \in F^{\otimes n}, \\
s_i x_j &= x_j s_i, && 1 \leq i \leq n-1,\ 1 \leq j \leq n,\ j \neq i, i+1, \\
s_i x_i &= x_{i+1} s_i - t_{i,i+1}, && 1 \leq i \leq n-1, \\
\pi \mathbf{f} &= {}^{\pi}\mathbf{f} \pi, && \pi \in S_n,\ \mathbf{f} \in F^{\otimes n},
\end{aligned}$$

where

$$t_{i,j} := \textstyle\sum_{b \in B} b_i b_j^\vee \quad \text{for} \quad 1 \leq i, j \leq n,\ i \neq j.$$

Affine wreath product algebras: Definition

The degree and parity on $\mathcal{A}_n(F)$ are determined by

$$|x_i| = \delta, \quad \bar{x}_i = 0, \qquad 1 \le i \le n,$$
$$|\pi| = 0, \quad \bar{\pi} = 0, \qquad \pi \in S_n,$$

while degree and parity for elements of $F^{\otimes n}$ are as they are in $F^{\otimes n}$.

Lemma (S. 2017)

Up to isomorphism, $\mathcal{A}_n(F)$ depends only on the underlying algebra F, and not on the trace map tr.

Proof: Uses fact that different trace maps differ by multiplication by an invertible element of F.

Affine wreath product algebras: Examples

Example ($F = \Bbbk$)

$\mathcal{A}_n(\Bbbk)$ is the degenerate affine Hecke algebra.

Example ($F = \mathrm{Cl}$)

$\mathcal{A}_n(\mathrm{Cl})$ is the affine Sergeev algebra, otherwise known as the degenerate affine Hecke–Clifford algebra.

This algebra was introduced by Nazarov in his study of the projective representation theory of the symmetric group.

Example ($F = \Bbbk G$)

$\mathcal{A}_n(\Bbbk G)$ is the wreath Hecke algebra studied by Wan and Wang.

Example (Affine zigzag algebras)

When F is a certain skew-zigzag algebra, $\mathcal{A}_n(F)$ is related to imaginary strata for quiver Hecke algebras by work of Kleshchev and Muth.

Structure theory

Deformed divided difference operators

Let

$$P_n = \Bbbk[x_1, \dots, x_n],$$

and let

$$P_n(F) = P_n \otimes F^{\otimes n},$$

where the two factors are subalgebras and

$$\mathbf{f} x_i = x_i \psi_i(\mathbf{f}), \qquad 1 \le i \le n,\ \mathbf{f} \in F^{\otimes n}.$$

Definition

Define a skew derivation $\Delta_i \colon P_n(F) \to P_n(F)$ inductively, by

$$\Delta_i(F^{\otimes n}) = 0,$$

$$\Delta_i(x_i) = t_{i,i+1}, \quad \Delta_i(x_{i+1}) = -t_{i+1,i}, \quad \Delta_i(x_j) = 0, \quad j \neq i, i+1,$$

and

$$\Delta_i(a_1 a_2) = \Delta_i(a_1) a_2 + {}^{s_i} a_1 \Delta_i(a_2), \quad a_1, a_2 \in P_n(F).$$

Deformed divided difference operators

Lemma (S. 2017)

For all $a \in P_n(F)$ and $1 \le i \le n-1$, in $\mathcal{A}_n(F)$ we have

$$s_i a = {}^{s_i}a\, s_i - \Delta_i(a).$$

The Δ_i are F-deformations of divided difference operators. In particular, if

$$\partial_i(p) = \frac{p - {}^{s_i}p}{x_i - x_{i+1}}, \quad p \in P_n,$$

is the usual divided difference operator, then

$$\Delta_i\left(x_i^k\right) = \sum_{b \in B} b_i \partial_i\left(x_i^k\right) b_{i+1}^\vee,$$

$$\Delta_i\left(x_{i+1}^k\right) = \sum_{b \in B} b_{i+1} \partial_i\left(x_{i+1}^k\right) b_i^\vee.$$

Deformed divided difference operators

The Δ_i have other properties analogous to those of divided difference operators.

Proposition (S. 2017)

We have

$$\begin{aligned}
\Delta_i\left({}^{s_j}a\right) &= {}^{s_j}\Delta_i(a), && 1 \le i,j \le n-1,\ |i-j|>1,\ a \in P_n(F),\\
\Delta_i\left({}^{s_i}a\right) &= -\,{}^{s_i}\Delta_i(a), && 1 \le i \le n-1,\ a \in P_n(F),\\
\Delta_i\Delta_j &= \Delta_j\Delta_i, && 1 \le i,j \le n-1,\ |i-j|>1,\\
\Delta_i^2 &= 0, && 1 \le i \le n-1.
\end{aligned}$$

Basis Theorem

Theorem (S. 2017)

The map

$$\Bbbk[x_1, \dots, x_n] \otimes F^{\otimes n} \otimes \Bbbk S_n \to \mathcal{A}_n(F),$$
$$p \otimes \mathbf{f} \otimes \pi \mapsto p\mathbf{f}\pi,$$

is an isomorphism of $\mathbb{Z} \times \mathbb{Z}_2$-graded $\Bbbk$-modules.

For special choices of F, recovers known results, but with a uniform proof.

Corollary

As $\mathbb{Z} \times \mathbb{Z}_2$-graded $\Bbbk$-modules, we have

$$\mathcal{A}_n(F) = \Bbbk[x_1, \dots, x_n] \otimes (F^{\otimes n} \rtimes S_n),$$

and the two factors are subalgebras.

This is the motivation for the name affine wreath product algebra.

The center

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},\ \alpha \in \mathbb{N}^n,$$

$$\mathbf{F}^{(\alpha)} = \left\{ \mathbf{f} \in F^{\otimes n} \mid \mathbf{g}\mathbf{f} = (-1)^{\bar{\mathbf{g}}\bar{\mathbf{f}}} \mathbf{f} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}(\mathbf{g}) \text{ for all } \mathbf{g} \in F^{\otimes n} \right\},$$

$$\mathbf{F}_\psi^{(\alpha)} = \{ \mathbf{f} \in \mathbf{F}^{(\alpha)} \mid \psi_i(\mathbf{f}) = \mathbf{f},\ 1 \le i \le n \}.$$

Theorem (S. 2017)

The center of $\mathcal{A}_n(F)$ consists of those elements of the form

$$\sum_{\alpha \in \mathbb{N}^n} x^\alpha \mathbf{f}_\alpha, \quad \mathbf{f}_\alpha \in \mathbf{F}_\psi^{(-\alpha)},$$

such that

$$\mathbf{f}_{\pi \cdot \alpha} = {}^\pi \mathbf{f}_\alpha \quad \text{for all } \alpha \in \mathbb{N}^n,\ \pi \in S_n.$$

The center: Examples

Example ($F = \mathbb{k}$)

We recover the well-known result that the center of the degenerate affine Hecke algebra consists of all symmetric polynomials in the x_i:

$$Z(\mathcal{A}_n(\mathbb{k})) = \mathbb{k}[x_1, \dots, x_n]^{S_n}.$$

Example ($F = \mathbb{k}G$)

We recover a description of the center of wreath Hecke algebras given by Wan and Wang.

Example ($F = \mathrm{Cl}$)

We recover the known result (due to Nazarov) that the center of the affine Sergeev algebra is

$$Z(\mathcal{A}_n(\mathrm{Cl})) = \mathbb{k}\left[x_1^2, \dots, x_n^2\right]^{S_n}.$$

The center: Corollary

Corollary

The center contains

$$\Bbbk\left[x_1^\theta, \dots, x_n^\theta\right]^{S_n},$$

where θ is the order of the Nakayama automorphism ψ.

In particular, $\mathcal{A}_n(F)$ is finitely generated as a module over its center.

Corollary

If $\Bbbk$ is an algebraically closed field, all simple $\mathcal{A}_n(F)$-modules are finite-dimensional.

Jucys–Murphy elements: The classical case ($\Bbbk S_n$)

In $\Bbbk S_n$, the Jucys–Murphy elements are

$$J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} (i,k), \quad 2 \le k \le n,$$

where $(i,k) \in S_n$ is the transposition of i and k.

Properties

1. We have a surjective algebra homomorphism

   $$\text{deg. aff. Hecke alg. } \mathcal{H}_n \twoheadrightarrow \Bbbk S_n$$

   that is the identity on $\Bbbk S_n$ and maps x_k to J_k.
2. J_n generates the centralizer of $\Bbbk S_{n-1}$ in $\Bbbk S_n$.
3. The center of $\Bbbk S_n$ consists of symmetric polynomials in the J_k.
4. Irreducible reps of S_n have bases given by standard tableaux of a given shape. Entries in the tableaux correspond to eigenvalues of the JM elements.

Juycs–Murphy elements for $\mathcal{A}_n(F)$

Define the Jucys–Murphy elements

$$J_1 = 0, \quad J_k = \sum_{i=1}^{k-1} t_{i,k}(i,k) = \sum_{i=1}^{k-1} \sum_{b \in B} b_i b_k^\vee (i,k), \quad 2 \le k \le n.$$

Proposition (S. 2017)

There is a surjective algebra homomorphism

$$\mathcal{A}_n(F) \twoheadrightarrow F^{\otimes n} \rtimes S_n,$$
$$x_k \mapsto J_k,\ 1 \le k \le n, \quad a \mapsto a,\ a \in F^{\otimes n} \rtimes S_n.$$

Examples

1. $F = \Bbbk$: the J_k are the usual JM elements for $\Bbbk S_n$.
2. $F = \mathrm{Cl}$: the J_k are the JM elements of the Sergeev algebra.
3. $F = \Bbbk G$: the J_k are the JM elements def. by Pushkarev and Wang.

Representation theory

Some associated algebras

Twisting by the Nakayama automorphism ψ yields a permutation of the isomorphism classes of irreducible representations of F.

Fix a list

$$\mathsf{L}_1, \mathsf{L}_2, \dots, \mathsf{L}_N$$

of representatives of the orbits of this permutation (up to degree shift).

For $1 \le \ell \le N$ and $n \in \mathbb{N}$, we let

$$\mathcal{H}_n^\ell = \begin{cases} \mathcal{A}_n(\Bbbk), \\ \mathcal{A}_n(\mathrm{Cl}), \\ (\Bbbk[y] \ltimes A)^{\otimes n} \rtimes S_n. \end{cases}$$

where the particular case depends on the representation L_ℓ, and A is a certain f.d. algebra related to L_ℓ.

Important: Simple modules of the $\mathcal{H}_n^\ell$ are classified.

An equivalence of categories

Let

$$\mathcal{R}_n = \bigoplus_{\mu} \left(\mathcal{H}^1_{\mu_1} \otimes \cdots \otimes \mathcal{H}^N_{\mu_N} \right),$$

where the sum is over all compositions of n of length at most N:

$$\mu = (\mu_1, \dots, \mu_N), \quad \sum \mu_i = n.$$

Theorem (S. 2017)

The category $\mathcal{R}_n$-mod is equivalent to the category of $\mathcal{A}_n(F)$-modules that are semisimple as $F^{\otimes n}$-modules.

Proposition (S. 2017)

Every simple $\mathcal{A}_n(F)$-module is semisimple as an $F^{\otimes n}$-module.

Classification of simple $\mathcal{A}_n(F)$-modules

Define the parabolic subalgebra

$$\mathcal{A}_\mu(F) = \mathcal{A}_{\mu_1}(F) \otimes \cdots \otimes \mathcal{A}_{\mu_N}(F) \subseteq \mathcal{A}_n(F).$$

Theorem (S. 2017)

Every simple $\mathcal{A}_n(F)$-module is isomorphic one of the form

$$\operatorname{Ind}_{\mathcal{A}_\mu(F)}^{\mathcal{A}_n(F)} \left(\mathsf{L}(\mu) \otimes_{\mathsf{E}(\mu)} (V_1 \circledast \cdots \circledast V_N)\right), \quad \text{where}$$

- μ is a composition of n of length at most N,
- V_j is a simple $\mathcal{H}^\ell_{\mu_\ell}$-module for $1 \leq \ell \leq N$,
- $\circledast$ denotes the simple tensor product of supermodules (simple summand of the usual tensor product),
- $\mathsf{E}(\mu)$ is a fixed subalgebra of $\mathcal{R}_\mu$,
- $\mathsf{L}(\mu)$ is a fixed bimodule.

Cyclotomic quotients

Cyclotomic quotients: The classical case

Any f.d. representation of the degenerate affine Hecke algebra factors through a f.d. cyclotomic quotient

$$\mathcal{H}_n^f := \mathcal{H}_n/(f(x_1)),$$

where f is a monic polynomial.

Induction/restriction functors on the categories $\mathcal{H}_n^f$-mod, $n \in \mathbb{N}$, relate these categories to the irreducible highest weight representation of an affine Lie algebra of type A.

The highest weight of the Lie algebra representation is encoded in the polynomial f.

This gives a powerful technique for studying the representation theory of $\mathcal{H}_n$ and its cyclotomic quotients.

Cyclotomic quotients for $\mathcal{A}_n(F)$

Recall that θ is the order of the Nakayama automorphism ψ and δ is the top degree of F.

For $1 \le k \le \theta$, let $\mathbf{F}_1^{(k)} \subseteq F^{\otimes n}$ consist of all elements $\mathbf{f} \in F^{\otimes n}$ such that

- $\psi_i(\mathbf{f}) = \mathbf{f}$ for all $1 \le i \le n$,
- $\mathbf{g}\mathbf{f} = \mathbf{f}\psi_1^k(\mathbf{g})$ for all $\mathbf{g} \in F^{\otimes n}$,
- ${}^\pi\mathbf{f} = \mathbf{f}$ for all $\pi \in S_n$ such that $\pi(1) = 1$.

Intuitively: $\mathbf{F}_1^{(k)}$ is the subspace of $F^{\otimes n}$ consisting of those elements that commute with elements of $\mathcal{A}_n(F)$ such as x_1^k does.

For $1 \le k \le \theta$, choose $e_k \in \mathbb{N}$ and degree $k\delta$ elements

$$\mathbf{c}^{(k,1)}, \dots, \mathbf{c}^{(k,e_k)} \in \mathbf{F}_1^{(k)}.$$

Define

$$\mathbf{C} = \left(\mathbf{c}^{(1,1)}, \dots, \mathbf{c}^{(1,e_1)}, \dots, \mathbf{c}^{(\theta,1)}, \dots, \mathbf{c}^{(\theta,e_\theta)}\right).$$

Cyclotomic quotients for $\mathcal{A}_n(F)$

Let $J_{\mathbf{C}}$ be the two-sided ideal in $\mathcal{A}_n(F)$ generated by

$$\chi_{\mathbf{C}} = \prod_{k=1}^{\theta} \prod_{j=1}^{e_k} \left(x_1^k - \mathbf{c}^{(k,j)} \right).$$

The cyclotomic wreath product algebra is

$$\mathcal{A}_n^{\mathbf{C}}(F) = \mathcal{A}_n(F)/J_{\mathbf{C}}.$$

The level $d_{\mathbf{C}}$ of $\mathbf{C}$ and $\mathcal{A}_n^{\mathbf{C}}(F)$ is the degree of $\chi_{\mathbf{C}}$ as a polynomial in x_1.

Note that the $\mathcal{A}_n^{\mathbf{C}}(F)$ are finite-dimensional.

Cyclotomic basis theorem

Theorem (S. 2017)

The canonical images of the elements

$$x^\alpha \mathbf{b}\pi, \quad \alpha_1, \dots, \alpha_n < d_{\mathbf{C}},\ \mathbf{b} \in B^{\otimes n},\ \pi \in S_n,$$

form a basis for $\mathcal{A}_n^{\mathbf{C}}(F)$.

Corollaries

1. $F = \Bbbk$ or $F = \mathrm{Cl}$: Theorem recovers known results.
2. $F = \Bbbk G$: Theorem recovers result of Wan and Wang.
3. F even and symmetric: Theorem proves an open conjecture of Kleshchev and Muth.

Corollary

Every level one cyclotomic wreath product algebra is isomorphic to the wreath product algebra $F^{\otimes n} \rtimes S_n$.

Frobenius algebra structure on $\mathcal{A}_n^{\mathbf{C}}(F)$

Define an even linear map $\operatorname{tr}_{\mathbf{C}} \colon \mathcal{A}_n^{\mathbf{C}}(F) \to \Bbbk$ by

$$\operatorname{tr}_{\mathbf{C}}\left(x^\alpha \mathbf{f} \pi\right) = \delta_{\alpha,(d-1,\dots,d-1)} \operatorname{tr}^{\otimes n}(\mathbf{f}) \delta_{\pi,1},$$

for $\mathbf{f} \in F^{\otimes n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\alpha_1, \dots, \alpha_n < d_{\mathbf{C}}$.

Theorem (S. 2017)

$\mathcal{A}_n^{\mathbf{C}}(F)$ is an $\mathbb{N}$-graded Frobenius superalgebra with trace map $\operatorname{tr}_{\mathbf{C}}$ and Nakayama automorphism given by

$$x_i \mapsto x_i, \quad \mathbf{f} \mapsto \left(\psi^{d_{\mathbf{C}}}\right)^{\otimes n}(\mathbf{f}), \quad \pi \mapsto \pi, \qquad 1 \le i \le n,\ \mathbf{f} \in F^{\otimes n},\ \pi \in S_n.$$

Corollary

F even and symmetric: Theorem gives a proof of another open conjecture of Kleshchev and Muth.

Future directions

Future directions

q-deformations

Many special cases of affine wreath product algebras are degenerations of q-deformed versions:

Degeneration	q-deformation
deg. affine Hecke algebra	affine Hecke algebra
affine Sergeev algebra	affine Hecke–Clifford algebra
wreath Hecke algebra	affine Yokonuma–Hecke algebra

It would be interesting to construct a q-deformation of affine wreath product algebras.

Double affine versions

Are there natural double affine versions of wreath product algebras generalizing double affine Hecke algebras (Cherednik algebras) and their various degenerations?

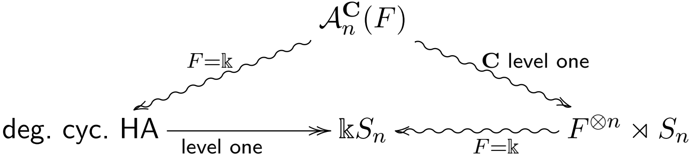
Future directions: Heisenberg categorification

Khovanov conjecturally categorified the Heisenberg algebra via a graphical category based on the rep theory of $\Bbbk S_n$. The degenerate affine Hecke algebra appears naturally in endomorphism spaces.

This was generalized by replacing $\Bbbk S_n$ by wreath product algebras (Rosso–S.). Affine wreath product algebras appear naturally in endomorphism spaces.

Khovanov's cat. was also generalized to higher level by replacing $\Bbbk S_n$ with degenerate cyclotomic Hecke algebras (Mackaay–S.).

There should be a graphical category based on $\mathcal{A}_n^{\mathbf{C}}(F)$ generalizing/unifying all of the above.



Future directions

Branching rules

- Branching rules involve explicit descriptions of restriction/induction functors acting on irreducible representations.
- For $F = \Bbbk, \mathrm{Cl},$ or $\Bbbk G$, branching rules are related to representation theory of affine Lie algebras.
- For F semisimple, previous methods should work.
- For general F, situation would be more involved.

Other types, formal group laws

- The $\mathcal{A}_n(F)$ are, in many ways, type A objects.
- It would be nice to generalize to other types.
- One can associate a formal affine Hecke algebra to any formal group law (Malagón-Lopez–Hoffnung–Zainouilline–S.).
- Can one define "Frobenius algebra deformations" of formal affine Hecke algebras?