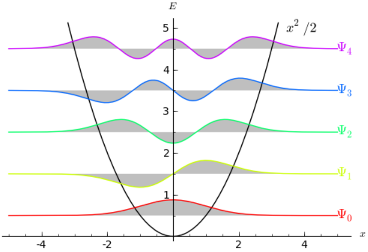
Towers of algebras categorify the Heisenberg double



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Outline

Summary:

- Modules over towers of algebras categorify the Heisenberg double.
- Examples: Heisenberg algebra, Weyl algebra, quasi-Heisenberg algebra.
- Application: The ring of quasisymmetric functions is free over the ring of symmetric functions (new proof).

Overview

1. What is categorification?
2. Heisenberg double
3. Towers of algebras
4. Categorification of the Heisenberg double
5. Examples/applications
6. The quasi-Heisenberg algebra and applications to quasisymmetric functions

What is categorification?

Suppose M is a module for a ring R.

We would like to find an abelian category $\mathcal{M}$ such that

$$\mathcal{K}_0(\mathcal{M}) \xrightarrow[\cong]{\varphi} M \qquad \text{(as } \mathbb{Z}\text{-modules)},$$

where $\mathcal{K}_0(\mathcal{M})$ is the Grothendieck group of $\mathcal{M}$.

Then, for each $r \in R$ (or, for those r in a fixed generating set), we want an exact endofunctor F_r of $\mathcal{M}$ such that we have a commutative diagram:

$$\begin{tikzcd} \mathcal{K}_0(\mathcal{M}) \arrow[r, "{[F_r]}"] \arrow[d, "\varphi"'] & \mathcal{K}_0(\mathcal{M}) \arrow[d, "\varphi"] \\ M \arrow[r, "r"] & M \end{tikzcd}$$

Here $[F_r]$ denotes the map induced by F_r on $\mathcal{K}_0(\mathcal{M})$.

What is categorification?

We would also like isomorphisms of functors lifting the relations of R.

For example, suppose we have a relation in R:

$$rs = 2sr + 3.$$

Then we would like isomorphisms of functors

$$F_r \circ F_s \cong (F_s \circ F_r)^{\oplus 2} \oplus \mathrm{Id}^{\oplus 3}.$$

Fruits of categorification

- Classes of objects (simple, indecomposable projective) give distinguished bases with positivity and integrality properties.
- Uncovers hidden structure in the algebra and its representation.
- Provides tools for studying the category $\mathcal{M}$.
- Applications to topology.

Notation

Fix a commutative ring $\Bbbk$. All Hopf algebras, etc. are over this ring.

For a Hopf algebra we have:

- ∇ – multiplication
- Δ – comultiplication
- η – unit
- ε – counit

We use Sweedler notation for coproducts:

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

Graded connected Hopf algebras

Definition (Graded connected Hopf algebra)

A bialgebra H is graded connected if $H = \bigoplus_{n \in \mathbb{N}} H_n$, where

- each H_n is f.g. and free as a $\Bbbk$-module,
- $\nabla(H_k \otimes H_\ell) \subseteq H_{k+\ell}$ and $\Delta(H_k) \subseteq \bigoplus_{j=0}^{k} H_j \otimes H_{k-j}$ for all $k, \ell \in \mathbb{N}$,
- $\eta(\Bbbk) \subseteq H_0$ and $\varepsilon(H_k) = 0$ for $k > 0$,
- $H_0 = \Bbbk 1_H$.

These axioms imply that H is a Hopf algebra with invertible antipode and so we also call it a graded connected Hopf algebra.

Examples

1. Polynomial ring $\Bbbk[x]$.
2. Ring Sym of symmetric functions in variables $x_1, x_2, \dots$.
3. Ring QSym of quasisymmetric functions.
4. Ring NSym of noncommutative symmetric functions.

Dual pairs of Hopf algebras

Definition (Dual pair)

We say (H^+, H^-) is a dual pair of Hopf algebras if

- $H^\pm$ are graded connected Hopf algebras,
- we have a perfect Hopf pairing $\langle \cdot, \cdot \rangle \colon H^- \otimes H^+ \to \Bbbk$.

Via this pairing, we identify $H^\pm$ with the graded dual of $H^\mp$.

Any $a \in H^+$ defines ${}^L a \in \operatorname{End} H^+$ by left multiplication

Any $x \in H^-$ defines ${}^R x \in \operatorname{End} H^-$ by right multiplication.
Taking adjoints yields ${}^R x^* \in \operatorname{End} H^+$.

So we have injective $\Bbbk$-algebra homomorphisms

$$H^+ \hookrightarrow \operatorname{End} H^+, \quad a \mapsto {}^L a,$$
$$H^- \hookrightarrow \operatorname{End} H^+, \quad x \mapsto {}^R x^*.$$

Note: $H^- \hookrightarrow \operatorname{End} H^+$ gives the left regular action of H^- on H^+.

The Heisenberg double

Definition (Heisenberg double)

The Heisenberg double of H^+ is the algebra $\mathfrak{h} = \mathfrak{h}(H^+, H^-)$ given by:

- $\mathfrak{h} = H^+ \otimes H^-$ as $\Bbbk$-modules. We write $a \# x$ for $a \otimes x$, viewed as an element of $\mathfrak{h}$.
- Multiplication is given by:

$$(a\#x)(b\#y) = \sum_{(x)} a\, {}^R x^*_{(1)}(b) \# x_{(2)} y = \sum_{(x),(b)} \langle x_{(1)}, b_{(2)} \rangle a b_{(1)} \# x_{(2)} y.$$

Remark: The Heisenberg double is a twist of the Drinfeld quantum double by a right 2-cocycle (Lu, 1994).

Fock space

Definition (Fock space)

The algebra $\mathfrak{h}$ has a natural representation on H^+ given by

$$(a\#x)(b) = a\,{}^R x^*(b), \quad a, b \in H^+,\ x \in H^-.$$

This is called the lowest weight Fock space representation of $\mathfrak{h}$ and we denote it $\mathcal{F}$.

Stone–von Neumann type theorem

1. The representation $\mathcal{F}$ is faithful.
2. If $\Bbbk$ is a field, then $\mathcal{F}$ is irreducible.
3. Any representation of $\mathfrak{h}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}$.

Thus, we may identify $\mathfrak{h}$ as the subalgebra of End H^+ generated by

$$^L a,\ {}^R x^*, \quad a \in H^+,\ x \in H^-.$$

Examples of the Heisenberg double

Example (Weyl algebra)

- Graded Hopf algebra: $\mathbb{Q}[x]$
- Heisenberg double: Weyl algebra $\langle x, \partial \mid \partial x = x\partial + 1 \rangle$
- Fock space: Polynomial representation on $\mathbb{Q}[x]$, $x \mapsto {}^L x$, $\partial \mapsto \frac{\mathrm{d}}{\mathrm{d}x}$

Example (Heisenberg algebra)

- Graded Hopf algebra: Ring Sym of symmetric functions
- Heisenberg double: Infinite-dimensional Heisenberg algebra
- Fock space: Bosonic Fock space

Example (Quasi-Heisenberg algebra)

- Graded Hopf algebra: Ring QSym of quasisymmetric functions
- Heisenberg double: Quasi-Heisenberg algebra
- Fock space: Natural action on QSym

The Heisenberg algebra as a Heisenberg double

To get a presentation of the Heisenberg double, we choose generators for H^+ and H^-.

Then a presentation is given by:

- the relations between the chosen generators of H^+,
- the relations between the chosen generators of H^-,
- the commutation relations between the generators of H^+ and the generators of H^-.

The Heisenberg algebra is the Heisenberg double of Sym:

$$\mathfrak{h} = H^+ \# H^-, \quad H^+ = H^- = \mathrm{Sym}.$$

To give a presentation of $\mathfrak{h}$, we choose generators of H^+ and H^- (i.e. two sets of generators of Sym).

The Heisenberg algebra as a Heisenberg double

Power sums

The power sums generate Sym over $\mathbb{Q}$.

- Let $p_1, p_2, \ldots$ be the power sum functions in $H^+ = \mathrm{Sym}$.
- Let $p_1^*, p_2^*, \ldots$ be the power sum functions in $H^- = \mathrm{Sym}$.

We get a presentation of $\mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Q}$ where the relations are:

$$p_m p_n = p_n p_m, \quad p_m^* p_n^* = p_n^* p_m^*, \quad p_m^* p_n = p_n p_m^* + m\delta_{m,n}.$$

Integral form

The elementary and complete symmetric functions generate Sym over $\mathbb{Z}$.

- Let $e_1, e_2, \ldots$ be the elementary symmetric functions in $H^+ = \mathrm{Sym}$.
- Let $h_1^*, h_2^*, \ldots$ be the complete symmetric functions in $H^- = \mathrm{Sym}$.

We get a presentation of $\mathfrak{h}$ (over $\mathbb{Z}$) where the relations are:

$$e_m e_n = e_n e_m, \quad h_m^* h_n^* = h_n^* h_m^*, \quad h_m^* e_n = e_n h_m^* + e_{n-1} h_{m-1}^*.$$

Goal

Goal: To categorify Heisenberg doubles and their Fock space representations.

So we want:

- categories whose Grothendieck groups are isomorphic to Fock space (as $\mathbb{Z}$-modules),
- functors lifting the action of the Heisenberg double,
- isomorphisms of functors lifting the defining relations of the Heisenberg algebra.

Outline:

- Categorify the (graded connected) Hopf algebra H^+ and its dual H^-.
- Define the functors that categorify the Fock space representation on H^+.

Towers of algebras

Definition (Tower of algebras)

A (strong) tower of algebras is a graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$ over $\mathbb{C}$ such that:

1. Each A_n is a f.d. algebra (with "internal" multiplication) and $A_0 \cong \mathbb{C}$.
2. The multiplication $A_m \otimes A_n \to A_{m+n}$ is a homomorphism of algebras for all m, n.
3. Each A_{m+n} is a two-sided projective $(A_m \otimes A_n)$-module.
4. Induction is (twisted) right adjoint to restriction.
5. A Mackey-like isomorphism relating induction and restriction holds.

Examples

- Group algebras of symmetric groups: $A_n = \mathbb{C}\mathfrak{S}_n$.
- Hecke algebras at generic q or at roots of unity.
- 0-Hecke algebras.
- Nilcoxeter algebras.

Module categories and their Grothendieck groups

For an f.d. algebra B, let

- B-mod $=$ category of f.g. left B-modules,
- B-pmod $=$ category of f.g. projective left B-modules.

Let

$$G_0(B) = \text{Grothendieck group of } B\text{-mod},$$
$$K_0(B) = \text{Grothendieck group of } B\text{-pmod}.$$

There is a natural pairing

$$\langle \cdot, \cdot \rangle \colon K_0(B) \otimes G_0(B) \to \mathbb{Z}, \quad \langle [P], [M] \rangle = \dim_{\Bbbk} \operatorname{Hom}_B(P, M).$$

If B is a subalgebra of A, we have:

- Induction: If N is a B-module, $\mathrm{Ind}_B^A N := A \otimes_B N$ is an A-module.
- Restriction: If M is an A-module, $\mathrm{Res}_B^A M := \operatorname{Hom}_A(A, M)$ is a B-module.

Grothendieck groups of towers of algebras

Let A be a tower of algebras. Let

$$\mathcal{G}(A) = \bigoplus_{n\in\mathbb{N}} G_0(A_n) \quad \text{and} \quad \mathcal{K}(A) = \bigoplus_{n\in\mathbb{N}} K_0(A_n).$$

We have a pairing $\langle\cdot,\cdot\rangle \colon \mathcal{K}(A) \times \mathcal{G}(A) \to \mathbb{Z}$ given by

$$\langle [P],[M]\rangle = \begin{cases} \dim_{\mathbb{C}} \mathrm{Hom}_{A_n}(P,M) & \text{if } P \in A_n\text{-pmod},\ M \in A_n\text{-mod},\\ 0 & \text{otherwise.}\end{cases}$$

We have a Hopf algebra structure on $\mathcal{G}(A)$ (or $\mathcal{K}(A)$) induced by

$$\nabla|_{(A_m\otimes A_n)\text{-mod}} \colon M \otimes N \mapsto \mathrm{Ind}_{A_m\otimes A_n}^{A_{m+n}}(M\otimes N),$$

$$\Delta|_{A_n\text{-mod}} \colon N \mapsto \bigoplus_{k+\ell=n} \mathrm{Res}_{A_k\otimes A_\ell}^{A_n} N,$$

$$\eta \colon \mathrm{Vect} \to A_0\text{-mod},\ V \mapsto V$$

$$\varepsilon|_{A_n\text{-mod}} \colon V \mapsto \begin{cases} V \in \mathrm{Vect} & \text{if } n = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Grothendieck groups of towers of algebras

The functors

$$\nabla, \quad \Delta, \quad \eta, \quad \varepsilon$$

are all exact. So they induce operators on the Grothendieck groups $\mathcal{G}(A)$ and $\mathcal{K}(A)$.

Proposition (cf. N. Bergeron–Li 2009)

Under the above operations, $(\mathcal{G}(A), \mathcal{K}(A))$ is a dual pair of Hopf algebras.

Distinguished bases

The classes of the simple objects give a distinguished basis of $\mathcal{G}(A)$.

The classes of the indecomposable projectives give a distinguished basis of $\mathcal{K}(A)$.

These bases have positive integral structure coefficients.

Examples

Example (Nilcoxeter algebras)

- $\mathcal{G}(A) = \mathbb{Z}[x]$, distinguished basis: x^n, $n \in \mathbb{N}$
- $\mathcal{K}(A) = \mathbb{Z}[x, x^2/2!, x^3/3!, \dots]$, distinguished basis: $x^n/n!$, $n \in \mathbb{N}$

Example (Groups algebras of symmetric groups)

- $\mathcal{G}(A) = \mathcal{K}(A) = \mathrm{Sym}$, distinguished basis: Schur functions

The same holds for Hecke algebras at a generic parameter.

Example (0-Hecke algebras)

- $\mathcal{G}(A) = \mathrm{QSym}$, distinguished basis: fundamental quasisymmetric functions
- $\mathcal{K}(A) = \mathrm{NSym}$, distinguished basis: noncommutative ribbon Schur functions

The Heisenberg double associated to a tower of algebras

Definition (Heisenberg double associated to a tower)

To a tower of algebras A, we can now associate the Heisenberg double

$$\mathfrak{h}(A) := \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A)) = \mathcal{G}(A) \# \mathcal{K}(A)$$

and its Fock space

$$\mathcal{F}(A) = \mathcal{G}(A).$$

Categorification of the Heisenberg double

We wish to categorify the Fock space representation of $\mathfrak{h}(A)$.

We have already categorified the underlying $\mathbb{Z}$-modules:

$$\mathcal{G}(A) = \bigoplus_{n \in \mathbb{N}} G_0(A_n).$$

We now need define functors on $\mathcal{G}(A)$ that lift the action of $\mathfrak{h}(A)$ on Fock space.

We would also like to have isomorphisms of functors that lift the defining relations of the Heisenberg double.

Categorification of the Heisenberg double

Consider the direct sum of categories:

$$A\text{-mod} = \bigoplus_{n \in \mathbb{N}} A_n\text{-mod}.$$

For $M \in A_m$-mod, we have the functor

$$\mathrm{Ind}_M \colon A\text{-mod} \to A\text{-mod},\ \mathrm{Ind}_M(N) = \mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}}(M \otimes N).$$

For $P \in A_p$-pmod, we have the functor

$$\mathrm{Res}_P \colon A\text{-mod} \to A\text{-mod},\ \mathrm{Res}_P(N) = \mathrm{Hom}_{A_p}(P, \mathrm{Res}_{A_{n-p} \otimes A_p}^{A_n} N).$$

These functors are exact and so they induce endomorphisms

$$[\mathrm{Ind}_M] \quad \text{and} \quad [\mathrm{Res}_P]$$

of $\mathcal{G}(A)$.

Categorification of the Heisenberg double

Theorem (S.–Yacobi 2013)

The functors Ind_M and Res_P for $M \in A$-mod and $P \in A$-pmod categorify the Fock space representation $\mathcal{F}(A)$ of $\mathfrak{h}(A)$.

More precisely, for all $M, N \in A$-mod and $P, Q \in A$-pmod, we have isomorphisms of functors

$$\mathrm{Ind}_M \circ \mathrm{Ind}_N \cong \mathrm{Ind}_{\nabla(M \otimes N)},$$
$$\mathrm{Res}_P \circ \mathrm{Res}_Q \cong \mathrm{Res}_{\nabla(P \otimes Q)},$$
$$\mathrm{Res}_P \circ \mathrm{Ind}_M \cong \nabla \mathrm{Res}_{\Delta(P)}(M \otimes -).$$

Thus, on the level of Grothendieck groups, we have

$$([M]\#[P])([N]) = [\mathrm{Ind}_M] \circ [\mathrm{Res}_P]([N]) = [\mathrm{Ind}_M \circ \mathrm{Res}_P(N)] \in \mathcal{G}(A).$$

Note: For simplicity, we have assumed that induction is biadjoint to restriction above (i.e. no twisting is needed).

Application: symmetric groups and the Heisenberg algebra

Let $\mathfrak{S}_n$ be the symmetric group on n letters.

Then $A = \bigoplus_{n \in \mathbb{N}} \Bbbk \mathfrak{S}_n$ is a tower of algebras.

We have

- $\mathcal{K}(A) \cong \mathcal{G}(A) \cong \mathrm{Sym}$ is the ring of symmetric functions,
- $\mathfrak{h}(A)$ is the classical (infinite rank) Heisenberg algebra,
- $\mathcal{F}(A)$ is the usual (bosonic) Fock space representation,
- Stone–von Neumann type theorem is the actual Stone–von Neumann Theorem.

Note: We get the same result if we replace $\Bbbk \mathfrak{S}_n$ by the Hecke algebra $H_n(q)$ for generic q.

So we get a categorification of the Heisenberg algebra. This recovers results of Geissinger and Zelevinsky.

Application: symmetric groups and the Heisenberg algebra

Distinguished basis of Fock space

The simple modules and the indecomposable projective modules are both the Spect modules.

These modules descend to the Schur functions in the Grothendieck group.

Categorification of relations

Recall that our integral form of the Heisenberg algebra had generators $\{e_n, h_n^*\}_{n \in \mathbb{N}_+}$ and relations

$$e_m e_n = e_n e_m, \quad h_m^* h_n^* = h_n^* h_m^*, \quad h_m^* e_n = e_n h_m^* + e_{n-1} h_{m-1}^*.$$

If E_n and L_n are the sign and trivial reps of $\mathfrak{S}_n$, respectively, then

$$\begin{gathered}
\mathrm{Ind}_{E_m} \circ \mathrm{Ind}_{E_n} \cong \mathrm{Ind}_{E_n} \circ \mathrm{Ind}_{E_m}, \\
\mathrm{Res}_{L_m} \circ \mathrm{Res}_{L_n} \cong \mathrm{Res}_{L_n} \circ \mathrm{Res}_{L_m}, \\
\mathrm{Res}_{L_m} \circ \mathrm{Ind}_{E_n} \cong \left(\mathrm{Ind}_{E_n} \circ \mathrm{Res}_{L_m}\right) \oplus \left(\mathrm{Ind}_{E_{n-1}} \circ \mathrm{Res}_{L_{m-1}}\right).
\end{gathered}$$

Application: symmetric groups and the Heisenberg algebra

So we see that we obtain a categorification of the presentation in terms of the $\{e_n, h_n^*\}_{n \in \mathbb{N}_+}$.

By considering other appropriate modules, we could get a categorification of any presentation. For example:

- generators $\{h_n, e_n^*\}_{n \in \mathbb{N}_+}$,
- generators $\{e_n, e_n^*\}_{n \in \mathbb{N}_+}$,
- generators $\{h_n, h_n^*\}_{n \in \mathbb{N}_+}$,
- etc.

The main theorems are presentation independent.

Application: nilcoxeter algebras and the Weyl algebra

Let N_n be the nilcoxeter algebra on n "strands".

Then $N = \bigoplus_{n \in \mathbb{N}} N_n$ is a tower of algebras.

We have

- $\mathcal{K}(N) \cong \mathbb{Z}[x]$, $\mathcal{G}(N) \cong \mathbb{Z}[x, x^2/2!, x^3/3!, \ldots]$,
- the element $x \in \mathcal{K}(N)$ acts on $\mathcal{G}(N)$ via ${}^R x^* = \partial_x$,
- $\mathfrak{h}(N)$ is the subalgebra of $\operatorname{End} \mathbb{Z}[x, x^2/2!, x^3/3!, \ldots]$ generated by $x, x^2/2!, x^3/3!, \ldots$ and ∂_x

Thus we get a categorification of the Weyl algebra and its polynomial representation. This recovers a result of Khovanov.

Application: nilcoxeter algebras and the Weyl algebra

Distinguished basis of Fock space

N_n has a unique simple module L_n of dimension one.

Its projective cover is N_n.

So we get distinguished bases:

$$\mathcal{G}(N) \cong \mathbb{Z}[x, x^2/!, x^3/3!, \dots], \quad [L_n] \mapsto x^n/n!,$$
$$\mathcal{K}(N) \cong \mathbb{Z}[x], \quad [N_n] \mapsto x^n.$$

Categorifiation of relations

If we let L be the trivial N_1-module, then we have

$$\mathrm{Res}_L \circ \mathrm{Ind}_L \cong (\mathrm{Ind}_L \circ \mathrm{Res}_L) \oplus \mathsf{Id}\,.$$

This is a categorification of the defining relation

$$\partial x = x\partial + 1.$$

Application: Hecke algebras at roots of unity

Let A_n be the Hecke algebra (of type A) specialized at an ℓ-th root of unity.

Then $A = \bigoplus_{n \in \mathbb{N}} A_n$ is a tower of algebras.

Let $\mathcal{J}_\ell \subseteq \mathrm{Sym}$ be the ideal generated by the power sums $p_\ell, p_{2\ell}, \ldots$.

We have

- $\mathcal{K}(A) \cong \mathcal{J}_\ell^\perp$ and $\mathcal{G}(A) \cong \mathrm{Sym}/\mathcal{J}_\ell$,
- $\mathfrak{h}(A)$ is an integral form of the Heisenberg algebra: $\mathfrak{h}(A) \otimes_\mathbb{Z} \mathbb{Q}$ is the Heisenberg algebra (over $\mathbb{Q}$),

Questions:

- Work out minimal representations of these integral forms.
- Is $\mathfrak{h}_{\mathrm{proj}}(A) \cong \mathfrak{h}(A)$?

Note: A-pmod categorifies the basic rep of $\widehat{\mathfrak{sl}}_n$ via i-induction and i-restriction. The above is a categorification of the action of the principal Heisenberg subalgebra of $\widehat{\mathfrak{sl}}_n$.

Quasisymmetric and noncommutative symmetric functions

Let QSym be the algebra of quasisymmetric functions.

Thus QSym is the subalgebra of $\mathbb{Z}[\![x_1, x_2, \ldots]\!]$ consisting of shift invariant functions.

That is, $f \in \mathrm{QSym}$ if and only if

$$\text{coeff of } x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \text{ in } f = \text{coeff of } x_{i_1}^{n_2} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k} \text{ in } f$$
$$\text{for all } 0 < i_1 < i_2 < \cdots < i_k \quad \text{and} \quad n_1, n_2, \ldots, n_k \in \mathbb{N}.$$

Let NSym be the algebra of noncommutative symmetric functions.

Thus NSym is the free associative algebra generated by $\mathbf{h}_1, \mathbf{h}_2, \ldots$.

Note: QSym and NSym are dual Hopf algebras.

The quasi-Heisenberg algebra

Let $H_n(0)$ be the 0-Hecke algebra on n "strands".

Then $A = \bigoplus_{n \in \mathbb{N}} H_n(0)$ is a tower of algebras and

$$\mathcal{K}(A) \cong \mathrm{NSym} \quad \text{and} \quad \mathcal{G}(A) \cong \mathrm{QSym}.$$

Definition (Quasi-Heisenberg algebra)

We call $\mathfrak{q} := \mathfrak{h}(A)$ the quasi-Heisenberg algebra.

Application: QSym is free over Sym

QSym is the Fock space for $\mathfrak{q}$.

Using an inductive argument involving the Stone–von Neumann type theorem, one can prove:

Theorem (Hazewinkel 2001, S.-Yacobi 2013)

The space QSym of quasisymmetric functions is free as a Sym-module.

Remark

The traditional proof of this result (due to Hazewinkel) involves constructing an explicit basis using modified Lyndon words.

The new proof is representation theoretic and follows from the Stone–von Neumann type theorem.

Future directions

Gradings

Many examples of categorification in the literature involve gradings (e.g. categories of graded modules over a graded algebra).

Often the passage from the non-graded to the graded setting results in a deformation of the algebraic object that is categorified.

Goal: Generalize the above results to the graded setting (ongoing work with Daniele Rosso).

Future directions

Strong/graphical categorification

The categorification of the Heisenberg algebra by Khovanov (q-deformed by Licata-S.) is stronger than the result obtained as a special case of the above main theorem (for the tower of symmetric groups).

Khovanov defines a graphical monoidal category whose Grothendieck group recovers the Heisenberg algebra.

Essentially, Khovanov's graphical category gives a precise description of the natural transformations involved in the isomorphisms of functors lifting the relations in the Heisenberg algebra.

Goal: Generalize this graphical approach to more general towers (ongoing work with Oded Yacobi).