Formal Hecke algebras and algebraic oriented cohomology theories

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Outline

Summary: Motivated by geometric constructions, we define two families of algebras depending on a formal group law associated to an algebraic oriented cohomology theory. These recover well-known algebras in certain cases and apparently new algebras in other cases.

Overview

Motivation – geometric constructions

Although our main result will be algebraic, the motivation comes from geometric constructions in representation theory.

Philosophy: We want to geometrize a representation $A \to \operatorname{End} V$ of some algebra A.

Algebra		Geometry
underlying vector space V	$\leftrightsquigarrow$	(co)homology of some variety
action of generators of A	$\leftrightsquigarrow$	geom. action (e.g. correspondences)
nice bases	$\leftrightsquigarrow$	classes of natural subvarieties

Advantages:

- often get bases with nice integrality and positivity properties
- geometrization can be a precursor to categorification

Motivation – examples

Many examples of these ideas are known (and have been discussed here).

Varieties		Algebra
Hilbert schemes	$\rightsquigarrow$	Heisenberg-Clifford algebras
quiver varieties	$\rightsquigarrow$	Kac-Moody algebras (or quantized versions)
(full) flag varieties	$\rightsquigarrow$	Hecke type algebras

Observation: Often, the algebra one obtains depends on the particular (co)homology theory used (e.g. singular cohomology, K-theory).

Motivating question: For a given variety, can one directly define a family of algebras where the (co)homology theory is an input and the output is the algebra obtained by using that cohomology theory in the geometric construction?

Today's focus: Flag varieties and Hecke type algebras.

Notation

For the rest of the talk, we fix a reduced root system with:

- weight lattice Λ with dual $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$,
- simple roots $\{\alpha_i \mid i \in I\} \subseteq \Lambda$,
- simple coroots $\{\alpha_i^\vee \mid i \in I\} \subseteq \Lambda^\vee$,
- pairing $\langle \cdot, \cdot \rangle$ between $\Lambda^\vee$ and Λ,
- reflections $\{s_i = s_{\alpha_i} \mid i \in I\}$, generating the Weyl group W.

Hecke algebra

Definition (Hecke algebra)

The (classical) Hecke algebra is the unital $\mathbb{Z}[t, t^{-1}]$-algebra H with

- generators: T_i, $i \in I$,
- quadratic relations: $(T_i + t^{-1})(T_i - t) = 0$ for all $i \in I$,
- braid relations: for all $i, j \in I$, with $s_i s_j$ of order m_{ij} in W,

$$\underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{factors}} = \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{factors}}.$$

Remarks

1. H is a t-deformation of the group algebra $\mathbb{Z}[W]$ of the Weyl group W.
2. Our conventions are different than found in some places in the literature:
   - $t = q^{1/2}$,
   - our tT_i corresponds to T_i in other presentation.

Affine Hecke algebra

Definition (Affine Hecke algebra)

The (classical) affine Hecke algebra is

- $H \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}][\Lambda]$ as a $\mathbb{Z}[t,t^{-1}]$-module,
- the factors H and $\mathbb{Z}[t,t^{-1}][\Lambda]$ are subalgebras,
- the relations between the factors are

$$e^\lambda T_i - T_i e^{s_i(\lambda)} = (t - t^{-1}) \frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}, \quad \lambda \in \Lambda,\ i \in I.$$

Here we write the group algebra as

$$\mathbb{Z}[t,t^{-1}][\Lambda] = \left\{ \textstyle\sum_{\lambda \in \Lambda} a_\lambda e^\lambda \mid a_\lambda \in \mathbb{Z}[t,t^{-1}] \right\}, \qquad e^\lambda e^{\lambda'} = e^{\lambda + \lambda'}.$$

Degenerate affine Hecke algebra

Definition (Degenerate affine Hecke algebra)

Let ϵ be an indeterminate. The degenerate affine Hecke algebra is the unital $\mathbb{Z}[\epsilon]$-algebra that is

- $\mathbb{Z}[W] \otimes_{\mathbb{Z}} S^*_{\mathbb{Z}[\epsilon]}(\Lambda)$ as a $\mathbb{Z}[\epsilon]$-module,
- the factors $\mathbb{Z}[W]$ and $S^*_{\mathbb{Z}[\epsilon]}(\Lambda)$ are subalgebras,
- the relations between the factors are

$$s_i \cdot \lambda - s_i(\lambda) \cdot s_i = -\epsilon \langle \alpha_i^\vee, \lambda \rangle, \quad i \in I,\ \lambda \in \Lambda.$$

Here

$$S^*_{\mathbb{Z}[\epsilon]}(\Lambda) = \bigoplus_{n=0}^{\infty} S^n_{\mathbb{Z}[\epsilon]}(\Lambda)$$

denotes the symmetric algebra of Λ over the ring $\mathbb{Z}[\epsilon]$.

Note: Often one sees the definition with $\epsilon = 1$.

"nil" Versions

0-Hecke algebra

Replace quadratic relations by $T_i(T_i + 1) = 0$ for all $i \in I$.

(Affine) nil Hecke algebra

Replace quadratic relation (in degenerate affine Hecke algebra) by $T_i^2 = 0$ for all $i \in I$ and set $\epsilon = 1$.

Refined motivating question #1

Many relations between these Hecke-type algebras are known. For example,

- the 0-Hecke algebra is the Hecke algebra at $q = 0$,
- the degenerate affine Hecke algebra is a certain limit (or graded version) of the affine Hecke algebra,
- the nil Hecke algebra is a certain limit of the 0-Hecke algebra.

Refined motivating question #1

Can one define some general algebras, depending on some sort of "input data", such that all of the above examples are simply special cases (corresponding to some choices of the input data)?

Geometric realizations

All of the algebras discussed above have geometric realizations.

We are interested in two particular geometric constructions:

1. "push-pull" operators on the cohomology of the flag variety, and
2. the convolution product on the cohomology of the Steinberg variety.

These geometric realizations will provide us with a clue as to what sort of "input data" we should consider.

The flag variety and push-pull operators

Let G be a split simple simply connected linear algebraic group over a field $\Bbbk$ corresponding to our root system.

T – split maximal torus

B – Borel subgroup containing T

G/B – variety of Borel subgroups of G

Example

If $\Bbbk = \mathbb{C}$ and $G = SL_n$, then

$$G/B \cong \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\}$$

is the (full) flag variety.

The flag variety and push-pull operators

simple root $\alpha_i \rightsquigarrow$ minimal parabolic subgroup P_i, with $B \subseteq P_i \subseteq G$.

We have the natural projection

$$p_i : G/B \twoheadrightarrow G/P_i,$$

and push and pull operators

$$(p_i)_* : \mathtt{h}(G/B) \to \mathtt{h}(G/P_i) \quad \text{and} \quad p_i^* : \mathtt{h}(G/P_i) \to \mathtt{h}(G/B).$$

Here $\mathtt{h}$ is any "suitable" cohomology theory, e.g., singular cohomology, K-theory (i.e. Grothendieck's K_0), etc.

Thus we have the push-pull operators

$$p_i^*(p_i)_* \in \operatorname{End} \mathtt{h}(G/B).$$

We also have $\mathtt{h}(G/B)$ acting on itself via left multiplication (cup product).

The flag variety and push-pull operators

The algebra generated by the push-pull operators and the left multiplication by $\mathtt{h}(G/B)$ depends on the cohomology theory $\mathtt{h}$.

Singular cohomology

- The push-pull operators generate the nil Hecke algebra.
- The push-pull operators and left multiplication generate the affine nil Hecke algebra.

K-theory

- The push-pull operators generate the 0-Hecke algebra.
- The push-pull operators and left multiplication generate the affine 0-Hecke algebra.

The fact that the nil Hecke algebra is a certain limit (or degeneration) of the 0-Hecke algebra can be interpreted geometrically using the Chern character map from K-theory to cohomology.

Convolution and the Steinberg variety

$\mathfrak{g}$ = Lie algebra of G

$\mathcal{N}$ = nilpotent cone of $\mathfrak{g}$ (i.e. set of all nilpotent elements of $\mathfrak{g}$)

$\widetilde{\mathcal{N}} = T^*(G/B)$ = cotangent bundle of G/B

There is a natural map

$$\mu : \widetilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}$$

which is a resolution of singularities called the Springer resolution.

Definition (Steinberg variety)

The Steinberg variety is the fiber product

$$Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \left\{ (x, y) \in \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}} \mid \mu(x) = \mu(y) \right\}.$$

Convolution and the Steinberg variety

There is a convolution product on $\mathtt{h}(Z)$, giving it the structure of an associative algebra.

Equivariant singular cohomology

- The convolution algebra is the degenerate affine Hecke algebra.

Equivariant K-theory

- The convolution algebra is the affine Hecke algebra.

Again, the fact that the degenerate affine Hecke algebra is a certain limit of the affine Hecke algebra can be interpreted in terms of the Chern character map from K-theory to singular cohomology.

Refined motivating question #2

Refined motivating question #2

The above algebras can be defined using any "suitable" cohomology theory. Rather than doing the procedure for each theory, can we give a uniform, purely algebraic definition with

- input: a cohomology theory
- output: an associative algebra

that predicts the algebra one obtains via the above geometric constructions?

In particular, inputting K-theory and singular cohomology should recover the algebras mentioned above.

Algebraic oriented cohomology theories

Algebraic oriented cohomology theory (AOCT): a contravariant functor $\mathtt{h}$ from the category of smooth projective varieties over a field $\Bbbk$ to the category of commutative unital rings which satisfies certain properties.

Examples:

- Chow groups, singular cohomology
- K-theory (Grothendieck's K_0)
- elliptic cohomology
- cobordism (universal AOCT)

To each AOCT is associated a formal group law which determines the first Chern class of a tensor product of two line bundles in terms of the first Chern classes of each line bundle.

This formal group law will be the input data for our algebras.

Formal group laws

Definition (Formal group law)

A (one-dimensional commutative) formal group law (FGL) is a pair (R, F) where

- R is a commutative domain (the coefficient ring),
- $F = F(u, v) \in R[\![u, v]\!]$ is a formal power series such that
  - $F(u, 0) = F(0, u) = u$,
  - $F(u, v) = F(v, u)$,
  - $F(u, F(v, w)) = F(F(u, v), w)$.

There is a unique formal inverse $-_F u \in R[\![u]\!]$ such that $F(u, -_F u) = 0$. It is divisible by u, and we let

$$\mu_F(u) := \frac{-_F u}{-u}.$$

Examples of FGLs

Example (Additive FGL)

The additive FGL

$$F_A(u, v) = u + v, \quad \mu_A(u) = 1$$

corresponds to Chow groups (and singular cohomology).

Example (Multiplicative FGL)

The multiplicative FGL

$$F_M(u, v) = u + v - \beta uv, \ \beta \in R, \ \beta \neq 0,$$
$$\mu_M(u) = \textstyle\sum_{i \geq 0} \beta^i u^i$$

corresponds to K-theory. If β is invertible in R, we call this the multiplicative periodic FGL.

Examples of FGLs

Example (Lorentz FGL)

The Lorentz FGL (addition of relativistic parallel velocities)

$$F_L(u,v) = \frac{u+v}{1+\beta uv} = (u+v)\sum_{i\geq 0}(-\beta uv)^i, \quad \beta \in R,\ \beta \neq 0$$

$$\mu_L(u) = 1$$

Example (Elliptic FGL)

The elliptic FGL F_E depends on a choice of elliptic curve E.

Example (Universal FGL)

The *Lazard ring* $\mathbb{L}$ is the commutative ring with generators a_{ij}, $i,j \in \mathbb{N}_+$, and subject to the relations that are forced by the axioms for FGLs. The corresponding FGL $(\mathbb{L}, F_U(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^i v^j)$ is called the universal FGL.

Formal group algebra

(R, F) a FGL

Λ an abelian group (e.g. our root lattice)

Let $R[x_\Lambda] := R[\{x_\lambda \mid \lambda \in \Lambda\}]$.

Augmentation map: $\varepsilon : R[x_\Lambda] \to R,\ x_\lambda \mapsto 0$ for all $\lambda \in \Lambda$

Let $R[\![\Lambda]\!]$ be the $(\ker \varepsilon)$-adic completion of $R[x_\Lambda]$.

Let J_F be the closure of the ideal of $R[\![\Lambda]\!]$ generated by

$$x_0 \quad \text{and} \quad \left(F(x_{\lambda_1}, x_{\lambda_2}) - x_{\lambda_1+\lambda_2}\right) \text{ for all } \lambda_1, \lambda_2 \in \Lambda.$$

Definition (Formal group algebra)

The formal group algebra is the quotient $R[\![\Lambda]\!]_F := R[\![\Lambda]\!]/J_F$.

Examples of formal group algebras

Suppose Λ is a free abelian group.

Example (Additive FGL)

$$R[\![\Lambda]\!]_A \cong S_R^*(\Lambda)^\wedge := \prod_{i=0}^{\infty} S_R^i(\Lambda)$$

Example (Multiplicative FGL)

$$R[\![\Lambda]\!]_M = R[\Lambda]^\wedge,$$

the $(\ker \epsilon)$-adic completion of the group algebra $R[\Lambda]$ of Λ, where ϵ is the augmentation map $e^\lambda \mapsto 1$.

Example

We have

$$R[\![\mathbb{Z}]\!]_A \cong R[\![\gamma]\!] \quad \text{and} \quad R[\![\mathbb{Z}]\!]_M \cong R[t, t^{-1}]^\wedge.$$

Twisted formal group algebra

Let Q denote the field of fractions of $R[\![\Lambda]\!]_F$.

The action of the Weyl group W on the root lattice induces

- an action on $R[\![\Lambda]\!]_F$, and hence
- an action on Q.

Let δ_w denote the element in $R[W]$ corresponding to w (so we have $\delta_{w'}\delta_w = \delta_{w'w}$ for $w, w' \in W$).

Definition (Twisted formal group algebra)

The twisted formal group algebra is the smash product

$$Q_W := R[W] \ltimes_R Q.$$

In other words, $Q_W = R[W] \otimes_R Q$ as an R-module, with multiplication determined by

$$(\delta_{w'}\psi')(\delta_w\psi) = \delta_{w'w} w^{-1}(\psi')\psi \text{ for all } w, w' \in W,\ \psi, \psi' \in Q.$$

Formal (affine) Demazure algebra

Definition (Formal Demazure element)

For $i \in I$, the corresponding formal Demazure element is

$$\Delta_i = \frac{1}{x_{\alpha_i}}(1 - \delta_{s_i}) \in Q_W.$$

This definition is motivated by Demazure operators.

Definition (Formal (affine) Demazure algebra)

The formal Demazure algebra D_F is the R-subalgebra of Q_W generated by the Δ_i.

The formal affine Demazure algebra $\mathbf{D}_F$ is the R-subalgebra of Q_W generated by D_F and $R[\![\Lambda]\!]_F$.

Formal (affine) Demazure algebra

Theorem (Malagón Lopez-Hoffnung-S.-Zainoulline '12)

The formal affine Demazure algebra $\mathbf{D}_F$ is generated by $R[\![\Lambda]\!]_F$ and Δ_i, $i \in I$, subject to the relations

1. $\varphi \Delta_i - \Delta_i s_i(\varphi) = \Delta_{\alpha_i}(\varphi)$ for all $i \in I$ and $\varphi \in R[\![\Lambda]\!]_F$;
2. $\Delta_i^2 = \Delta_i \kappa_i$ for all $i \in I$, where $\kappa_i = \frac{1}{x_{\alpha_i}} + \frac{1}{x_{-\alpha_i}} \in R[\![\Lambda]\!]_F$;
3. $\Delta_i \Delta_j = \Delta_j \Delta_i$ for all $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = 0$;
4. braid relations up to lower order terms for all $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle \neq 0$.

Here Δ_{α_i} is the formal Demazure operator

$$\Delta_{\alpha_i} \colon R[\![\Lambda]\!]_F \to R[\![\Lambda]\!]_F, \quad \Delta_{\alpha_i}(\varphi) = \frac{\varphi - s_i(\varphi)}{x_{\alpha_i}}.$$

Formal (affine) Demazure algebra

One can explicitly compute the “braid relations”. For example

$$\Delta_j \Delta_i \Delta_j - \Delta_i \Delta_j \Delta_i = \Delta_i \kappa_{ij} - \Delta_j \kappa_{ji}$$

for all $i, j \in I$ such that $s_i s_j$ has order three (e.g. adjacent nodes in type A), where

$$\kappa_{ij} = \frac{1}{x_{\alpha_i + \alpha_j}} \left(\frac{1}{x_{\alpha_j}} - \frac{1}{x_{-\alpha_i}} \right) - \frac{1}{x_{\alpha_i} x_{\alpha_j}} \in R[\![\Lambda]\!]_F.$$

Remark

The true braid relations (i.e. where the lower order terms are actually zero) are satisfied only for the additive and multiplicative FGLs.

Additive and multiplicative cases

Special case: Additive FGL

For the additive FGL (over $\mathbb{Z}$),

- D_A is the (completion of the) nil Hecke algebra (no poly. part),
- $\mathbf{D}_A$ is the (completion of the) affine nil Hecke algebra.

Special case: Multiplicative FGL

For the multiplicative periodic FGL (over $\mathbb{Z}$),

- D_M is the (completion of the) 0-Hecke algebra,
- $\mathbf{D}_M$ is the (completion of the) affine 0-Hecke algebra.

Other cases

For other FGLs, the (affine) Demazure algebras appear to be new.
Example: For the Lorentz FGL, the "braid relation" becomes

$$\Delta_j \Delta_i \Delta_j - \Delta_i \Delta_j \Delta_i = \beta(\Delta_i - \Delta_j) \quad \text{for } s_i s_j \text{ of order 3.}$$

Formal (affine) Hecke algebra

We will modify the construction by introducing a group $\Gamma \cong \mathbb{Z}$ with generator γ.

We change the coefficient ring. Let $R_F := R[\![\Gamma]\!]_F$. For example:

- $\mathbb{Z}_A = \mathbb{Z}[\![\gamma]\!]$,
- $\mathbb{Z}_M = \mathbb{Z}[t, t^{-1}]^\wedge$.

Let Q' be the fraction field of $R_F[\![\Lambda]\!]_F$.

Let $Q'_W := R_F[W] \ltimes Q'$ be the corresponding twisted formal group algebra over R_F.

Let $\Theta = \mu_F(x_\gamma) - \mu_F(x_{-\gamma}) \in R_F$.

Note: $\mu_F(x_\gamma)$ will play the role of the deformation parameter t in the usual Hecke algebra.

Simplifying assumption: For the purposes of this talk, we assume that either $F = F_A$ or the coefficient of uv in $F(u, v)$ is invertible.

Formal (affine) Hecke algebra

Recall that, for $i \in I$, we have

$$\kappa_i = \frac{1}{x_{\alpha_i}} + \frac{1}{x_{-\alpha_i}} \in R_F[\![\Lambda]\!]_F.$$

For $i \in I$, let

$$T_i := \begin{cases} \Delta_i \frac{\Theta_F}{\kappa_i} + \delta_{s_i} \mu(x_\gamma) & \text{if } \mu_F \neq 1, \\ 2\Delta_i x_\gamma + \delta_{s_i} & \text{if } \mu_F = 1. \end{cases}$$

Definition (Formal (affine) Hecke algebra)

The formal Hecke algebra H_F is the R_F-subalgebra of Q'_W generated by the T_i, $i \in I$.

The formal affine Hecke algebra $\mathbf{H}_F$ is the R_F-subalgebra of Q'_W generated by H_F and $R_F[\![\Lambda]\!]_F$.

Formal (affine) Hecke algebra

Theorem (Malagón Lopez-Hoffnung-S.-Zainoulline '12)

The formal affine Hecke algebra $\mathbf{H}_F$ satisfies the following relations:

1. $\varphi T_i - T_i s_i(\varphi) = \begin{cases} \frac{\Theta_F}{\kappa_i} \Delta_{\alpha_i}(\varphi) & \text{if } \mu_F \neq 1, \\ 2x_\gamma \Delta_{\alpha_i}(\varphi) & \text{if } \mu_F = 1, \end{cases} \quad \forall\ i \in I,\ \varphi \in R_F[\![\Lambda]\!]_F.$
2. $(T_i + \mu_F(x_{-\gamma}))(T_i - \mu_F(x_\gamma)) = 0$ for all $i \in I$,
3. $T_i T_j = T_j T_i$ for all $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = 0$,
4. braid relations up to lower order terms for all $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle \neq 0$.

These form a complete set of relations over a slightly enlarged coefficient ring.

As for the formal (affine) Demazure algebra, one can explicitly compute the "braid relations". The true braid relations only hold for the additive and multiplicative FGLs.

Additive and multiplicative cases

Special case: Additive FGL

For the additive FGL (over $\mathbb{Z}$),

- $H_A = \mathbb{Z}_A[W]$ is group algebra of the Weyl group,
- $\mathbf{H}_A$ is the (completion of the) degenerate affine Hecke algebra.

Special case: Multiplicative FGL

For the multiplicative periodic FGL (over $\mathbb{Z}$),

- H_M is the (completion of the) Hecke algebra,
- $\mathbf{H}_M$ is the (completion of the) affine Hecke algebra.

Other cases

For other FGLs, the (affine) Hecke algebras appear to be new.
Example: For the Lorentz FGL, the "braid relation" becomes

$$\Delta_j \Delta_i \Delta_j - \Delta_i \Delta_j \Delta_i = 4\beta x_\gamma^2 (\Delta_i - \Delta_j) \qquad \text{for } s_i s_j \text{ of order 3.}$$

Motivating questions revisited

Question 1: Answered

We have answer the refined motivating question #1.

Namely, we have defined families of algebras that specialize to the Hecke type algebras we were interested in.

Question 2

Do these families answer the second question? I.e. do they predict the algebras one obtains from the geometric constructions?

- For cohomology and K-theory, the answer is yes.
- For other theories (i.e. not additive or multiplicative), some work needs to be done to define the convolution constructions for these theories.
- Once this is done, we expect that our algebras answer our second question in the affirmative.

Summary

Given a FGL (R, F), we have defined:

- the formal Demazure algebra and formal affine Demazure algebra,
- the formal Hecke algebra and formal affine Hecke algebra.

For the additive and multiplicative FGLs, we obtain important known algebras:

	Additive FGL	Multiplicative FGL
AOCT	(Equiv.) singular cohomology	(Equiv.) K-theory
FDA	Nil Hecke alg.	0-Hecke alg.
FADA	Affine nil Hecke alg.	Affine 0-Hecke alg.
FHA	Group alg. of the Weyl Group	Hecke alg.
FAHA	Degenerate affine Hecke alg.	Affine Hecke alg.

For other FGLs, we seem to obtain new algebras.

Further directions

Question: Can one define other families of algebras, depending on a FGL, for other geometric constructions?

- quiver varieties
- Hilbert schemes

Quiver varieties:

- equivariant cohomology $\rightsquigarrow$ Yangian
- equivariant K-theory $\rightsquigarrow$ quantum loop algebra

Recent work of Gautam and Toledano-Laredo relates the above algebras (extending work of Drinfel′d).

We expect that one can define a family of algebras in this case and obtain results similar to the ones presented today for Hecke algebras.