The representation theory of equivariant map algebras

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Joint work with

- Neher-Senesi (arXiv:0906.5189, to appear in TAMS)
- Neher (arXiv:1103.4367)
- Fourier-Khandai-Kus (to be posted this week)

Outline

Review: Classification of irreducible finite-dimensional representations of equivariant map algebras.

Goal: Describe extensions and block decompositions.

Overview:

1. Equivariant map algebras
2. Examples
3. Evaluation representations
4. Classification of finite-dimensional irreducibles
5. Extensions
6. Block decompositions
7. Weyl modules

Terminology:

small = irreducible finite-dimensional

(Untwisted) Map algebras

Notation

k - algebraically closed field of characteristic zero

X - scheme (or algebraic variety) over k

$A = A_X = \mathcal{O}_X(X)$ - coordinate ring of X

$\mathfrak{g}$ - finite-dimensional Lie algebra over k

Definition (Untwisted map algebra)

$M(X, \mathfrak{g})$ = Lie algebra of regular maps from X to $\mathfrak{g}$

Pointwise multiplication:

$$[\alpha, \beta]_{M(X,\mathfrak{g})}(x) = [\alpha(x), \beta(x)]_{\mathfrak{g}} \text{ for } \alpha, \beta \in M(X, \mathfrak{g})$$

Note: $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes A_X$

Examples

Discrete spaces

If X is a discrete variety, then

$$M(X, \mathfrak{g}) \cong \prod_{x \in X} \mathfrak{g}, \quad \alpha \mapsto (\alpha(x))_{x \in X}, \quad \alpha \in M(X, \mathfrak{g}).$$

In particular, if $X = \{x\}$ is a point, then

$$M(X, \mathfrak{g}) \cong \mathfrak{g}, \quad \alpha \mapsto (\alpha(x)), \quad \alpha \in M(X, \mathfrak{g}).$$

The isomorphisms are given by evaluation.

Current algebras

$$X = k^n \implies A_X = k[t_1, \ldots, t_n]$$

Thus, $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1, \ldots, t_n]$ is a current algebra.

Equivariant map algebras

- Γ - finite group
- Suppose Γ acts on X and $\mathfrak{g}$ by automorphisms

Definition (equivariant map algebra)

The equivariant map algebra is the Lie algebra of Γ-equivariant maps from X to $\mathfrak{g}$:

$$M(X, \mathfrak{g})^\Gamma = \{\alpha \in M(X, \mathfrak{g}) : \alpha(\gamma \cdot x) = \gamma \cdot \alpha(x)\ \forall\ x \in X,\ \gamma \in \Gamma\}$$

Note: If X is any scheme, then $M(X, \mathfrak{g})^\Gamma \cong M(X_{\mathrm{aff}}, \mathfrak{g})^\Gamma$ where $X_{\mathrm{aff}} = \operatorname{Spec} A_X$ is the affine scheme with the same coordinate ring as X. So we often assume X is affine.

Example: multiloop algebras

$$\Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}, \quad X = (k^\times)^n$$

- For $i = 1, \ldots, n$, let ξ_i be a primitive m_i-th root of unity.
- Define action of Γ on X by

$$(a_1, \ldots, a_n) \cdot (z_1, \ldots, z_n) = (\xi_1^{a_1} z_1, \ldots, \xi_n^{a_n} z_n)$$

- Define action of Γ on $\mathfrak{g}$ by specifying commuting automorphisms σ_i, $i = 1, \ldots, n$, such that $\sigma_i^{m_i} = 1$.

Then $M(X, \mathfrak{g})^\Gamma$ is the (twisted) multiloop algebra.

If $n = 1$, this is the (twisted) loop algebra.

Affine Lie algebras

The affine Lie algebras can be constructed as central extensions of loop algebras plus a differential:

$$\widehat{\mathfrak{g}} = M(X, \mathfrak{g})^\Gamma \oplus kc \oplus kd \qquad (n = 1)$$

Example: generalized Onsager algebra

$$\Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on X by $\sigma \cdot x = x^{-1}$
- Γ acts on $\mathfrak{g}$ by any involution

When Γ acts on $\mathfrak{g}$ by the Chevalley involution, we write

$$\mathcal{O}(\mathfrak{g}) = M(X, \mathfrak{g})^\Gamma$$

Remarks

- If $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to the Onsager algebra (Roan 1991)
  - Key ingredient in Onsager's original solution of the 2D Ising model
- For $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_n)$ was studied by Uglov and Ivanov (1996)

Evaluation

If $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq X$, we have the evaluation map

$$\mathrm{ev}_{\mathbf{x}} : M(X, \mathfrak{g})^\Gamma \to \mathfrak{g}^{\oplus n}, \quad \alpha \mapsto (\alpha(x_i))_i$$

Important: This map is not surjective in general!

For $x \in X$, define

$$\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}$$
$$\mathfrak{g}^x = \{u \in \mathfrak{g} : \Gamma_x \cdot u = u\}$$

Lemma

For X affine, $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq X$, $x_i \notin \Gamma \cdot x_j$ for $i \neq j$,

$$\mathrm{im}\, \mathrm{ev}_{\mathbf{x}} = \oplus_i \mathfrak{g}^{x_i}.$$

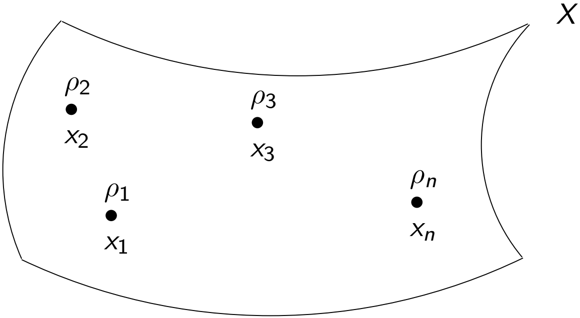
Evaluation representations

Given

- $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq X$, and
- representations $\rho_i : \mathfrak{g}^{x_i} \to \mathrm{End}_k V_i$, $i = 1, \ldots, n$,

we define the (twisted) evaluation representation as the composition

$$M(X, \mathfrak{g})^\Gamma \xrightarrow{\mathrm{ev}_\mathbf{x}} \oplus_i \mathfrak{g}^{x_i} \xrightarrow{\otimes_i \rho_i} \mathrm{End}_k(\otimes_i V_i).$$



Important remarks

This notion of evaluation representation differs from the classical definition.

- Some authors use the term evaluation representation only for the case when evaluation is at a single point and call the general case a tensor product of evaluation representations.
- To a point $x \in X$, we associate a representation of $\mathfrak{g}^x$ instead of $\mathfrak{g}$. If Γ acts freely, this coincides with the usual definition.
- Recall that (when $\mathfrak{g}^x \subsetneq \mathfrak{g}$) not all reps of $\mathfrak{g}^x$ extend to reps of $\mathfrak{g}$ – so the new definition is more general.

We will see that the more general definition allows for a more uniform classification of representations.

Evaluation representations

$$\mathcal{R}_x = \{\text{isomorphism classes of small reps of } \mathfrak{g}^x\}$$

$$\mathcal{R}_X = \bigsqcup_{x \in X} \mathcal{R}_x$$

We have an action of Γ on $\mathcal{R}_X$: if $[\rho] \in \mathcal{R}_x$, then

$$\gamma \cdot [\rho] = [\rho \circ \gamma^{-1}] \in \mathcal{R}_{\gamma \cdot x}.$$

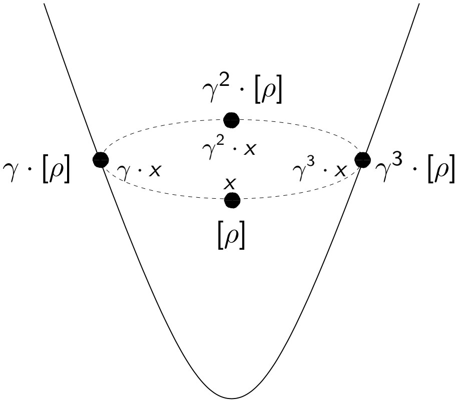
Definition ($\mathcal{E}$)

$\mathcal{E}$ is set of all $\psi : X \to \mathcal{R}_X$ such that

1. ψ is Γ-equivariant,
2. $\psi(x) \in \mathcal{R}_x$ for all $x \in X$, and
3. $\operatorname{supp} \psi = \{x \in X : \psi(x) \neq 0\}$ is finite.

Evaluation representations

We think of $\psi \in \mathcal{E}$ as assigning a finite number of (isom classes of) reps of $\mathfrak{g}^x$ to points $x \in X$ in a Γ-equivariant way.



Evaluation representations

For each $\psi \in \mathcal{E}$, define

$$\mathrm{ev}_\psi = \mathrm{ev}_{\mathbf{x}}(\psi(x_i))_{i=1}^n = \mathrm{ev}_{x_1}\psi(x_1) \otimes \cdots \otimes \mathrm{ev}_{x_n}\psi(x_n)$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of points of X containing one point from each Γ-orbit in $\operatorname{supp}\psi$ (the isom class is independent of this choice).

Lemma

For $\psi \in \mathcal{E}$, ev_ψ is the isomorphism class of a small representation of $M(X, \mathfrak{g})^\Gamma$.

Proposition

The map

$$\mathcal{E} \longrightarrow \{\text{isom classes of small reps of } M(X, \mathfrak{g})^\Gamma\}, \quad \psi \mapsto \mathrm{ev}_\psi,$$

is injective. In other words, $\mathcal{E}$ enumerates the small evaluation representations.

One-dimensional representations

Recall: Any 1-dimensional rep of a Lie algebra L corresponds to a linear map $\lambda : L \to k$ such that $\lambda([L, L]) = 0$.

We identify such 1-dimensional reps with elements

$$\lambda \in (L/[L, L])^*.$$

Two 1-dimensional reps are isomorphic if and only if they are equal as elements of $(L/[L, L])^*$.

Classification Theorem

Theorem (Neher-S.-Senesi 2009)

Suppose Γ *is a finite group acting on an affine scheme (or variety)* X *and a finite-dimensional Lie algebra* $\mathfrak{g}$. *Let* $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Then the map

$$(\lambda, \psi) \mapsto \lambda \otimes \mathrm{ev}_\psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \psi \in \mathcal{E}$$

gives a surjection

$$(\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^* \times \mathcal{E} \twoheadrightarrow \{\textit{isom classes of small representations of } \mathfrak{M}\}.$$

In particular, all small representations are of the form

$$(\textit{1-dim rep}) \otimes (\textit{evaluation rep}).$$

Classification – Remarks

$$(\lambda, \psi) \mapsto \lambda \otimes \mathrm{ev}_\psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*, \quad \psi \in \mathcal{E}$$

1. This map is not injective in general since we can have nontrivial evaluation reps which are 1-dimensional. This happens when $\mathfrak{g}^x$ is not perfect (e.g. reductive but not semisimple).

   Example: $\mathfrak{g} = \mathfrak{sl}_2$, $\Gamma = \mathbb{Z}_2$, $X = k = \mathbb{C}$
   - Γ acts on $\mathfrak{g}$ by the Chevalley involution.
   - Γ acts on X by multiplication by -1.
   - Then $\mathfrak{g}^0 = \mathfrak{g}^\Gamma$ is one-dimensional and so has nontrivial 1-dim reps.

2. However, we can specify precisely when $\lambda \otimes \mathrm{ev}_\psi \cong \lambda' \otimes \mathrm{ev}_{\psi'}$.

3. The restriction of the map to either factor is injective.

Classification

$$(\lambda, \psi) \mapsto \lambda \otimes \mathrm{ev}_\psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*, \quad \psi \in \mathcal{E}$$

Corollary

1. *If $\mathfrak{M}$ is perfect (i.e. $\mathfrak{M} = [\mathfrak{M},\mathfrak{M}]$), then we have a bijection*

$$\mathcal{E} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \psi \mapsto \mathrm{ev}_\psi .$$

   In particular, all small reps are evaluation reps.

2. *If $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{M}$ is perfect and the above bijection holds.*
3. *If Γ acts on $\mathfrak{g}$ by diagram automorphisms, then $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$ and the above bijection holds.*
4. *If Γ is abelian and acts freely, then $\mathfrak{M}$ is perfect and the above bijection holds.*

Note: Being perfect is not a necessary condition for all small reps to be evaluation reps (as we will see).

Application: untwisted map algebras

If Γ is trivial, then

$$M(X, \mathfrak{g})^\Gamma = M(X, \mathfrak{g}), \quad \mathfrak{g}^\Gamma = \mathfrak{g}.$$

Thus, if $\mathfrak{g}$ is perfect,

$$[\mathfrak{g}^\Gamma, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$$

and so all small reps are evaluation reps.

Application: multiloop algebras

Corollary

If $\mathfrak{M}$ is a (twisted) multiloop algebra, then $\mathfrak{M}$ is perfect and so we have a bijection

$$\mathcal{E} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \psi \mapsto \mathrm{ev}_\psi .$$

In particular, all small reps are evaluation reps.

Remarks

1. This recovers results of Chari-Pressley (for loop algebras) and Batra, Lau (multiloop algebras), but with a different description.
2. The description given above (in terms of $\mathcal{E}$) gives a simple and uniform description of the somewhat technical conditions appearing in previous classifications.
3. Action of Γ on X is free and so $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$. So the more general notion of evaluation rep does not play a role.

Application: generalized Onsager algebra

$$\Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on X by $\sigma \cdot x = x^{-1}$
- Γ acts on $\mathfrak{g}$ by any involution

Corollary

With Γ, X, $\mathfrak{g}$ as above, we have a bijection

$$\mathcal{E} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \psi \mapsto \mathrm{ev}_\psi\,.$$

In particular, all small reps are evaluation reps.

Remarks – generalized Onsager algebra

- There are two types of points of X:
  - $x \in \{\pm 1\} \implies \Gamma_x = \Gamma = \mathbb{Z}_2$, $\mathfrak{g}^x = \mathfrak{g}^\Gamma$
  - $x \notin \{\pm 1\} \implies \Gamma_x = \{1\}$, $\mathfrak{g}^x = \mathfrak{g}$
- $\mathfrak{g}^\Gamma$ can be semisimple or reductive with one-dimensional center

When $\mathfrak{g}^\Gamma$ has a one-dimensional center:

- the generalized Onsager algebra is not perfect
- we can place (nontrivial) one-dim reps of $\mathfrak{g}^\Gamma$ at the points ± 1
- under our more general definition of evaluation rep, all small reps are evaluation reps
- under classical notion of evaluation rep, there are small reps which are not evaluation reps

Moral: The more general definition of evaluation rep allows for a more uniform classification.

Special case: Onsager algebra

- When $k = \mathbb{C}$ and Γ acts on $\mathfrak{g} = \mathfrak{sl}_2$ by the Chevalley involution, then

$$\mathcal{O}(\mathfrak{sl}_2) \stackrel{\text{def}}{=} M(X, \mathfrak{sl}_2)^\Gamma$$

  is the Onsager algebra.

- $\mathfrak{g}^{\{\pm 1\}}$ is one-dimensional abelian and $\mathcal{O}(\mathfrak{sl}_2)$ is not perfect.
- Small reps of $\mathcal{O}(\mathfrak{sl}_2)$ were classified previously (Date-Roan 2000)
  - classical definition of evaluation rep was used
  - not all small reps were evaluation reps
  - this necessitated the introduction of the type of a representation

Note: For the other cases, the classification seems to be new.

Extensions

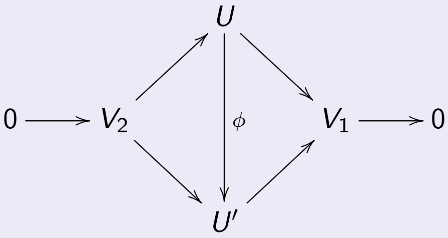
Suppose L is an arbitrary Lie algebra.

Definition (Extension)

An extension of an L-module V_1 by an L-module V_2 is a short exact sequence of L-modules

$$0 \to V_2 \to U \to V_1 \to 0.$$

Two extensions are equivalent if there is a map ϕ such that



is commutative.

Extensions

$\mathrm{Ext}^1_L(V_1, V_2) =$ the set of equivalence classes of extensions.

L semisimple

When L is semisimple, all finite-dimensional representations are completely reducible and hence

$$\mathrm{Ext}^1_L(V_1, V_2) = \{0\}.$$

Here (and always) 0 is the equivalence class of the trivial extension $V_1 \oplus V_2$.

Goal: Describe the extensions of small representations of equivariant map algebras.

Evaluation modules with disjoint support

Consider an EMA $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Suppose $\mathfrak{g}$ is reductive and A is finitely generated.

Proposition (Neher-S. 2011)

Suppose $\psi, \psi' \in \mathcal{E}$ *such that*

- $\operatorname{supp} \psi \cap \operatorname{supp} \psi' = \emptyset$, *and*
- ev_ψ *and* $\mathrm{ev}_{\psi'}$ *are nontrivial.*

Then

$$\mathrm{Ext}^1_{\mathfrak{M}}(\mathrm{ev}_\psi, \mathrm{ev}_{\psi'}) = 0.$$

Remark

In the case Γ is trivial, this was proven by Kodera (for current algebras, it was proven by Chari-Moura).

Extensions between evaluation modules

Theorem (Neher-S. 2011)

Suppose that V, V' are evaluation modules corresponding to $\psi, \psi' \in \mathcal{E}$. Let

$$V = \bigotimes_{x \in \mathbf{x}} V_x, \quad V' = \bigotimes_{x \in \mathbf{x}} V'_x$$

for a finite subset $\mathbf{x} \subseteq X$ that does not contain two points in the same orbit, and V_x, V'_x eval reps at x.

1. *If ψ, ψ' differ on more than one orbit, then $\mathrm{Ext}^1_{\mathfrak{M}}(V, V') = 0$.*
2. *If ψ, ψ' differ on exactly one orbit $\Gamma \cdot x_0$, then*

$$\mathrm{Ext}^1_{\mathfrak{M}}(V, V') \cong \mathrm{Ext}^1_{\mathfrak{M}}(V_{x_0}, V'_{x_0}).$$

3. *If $\psi = \psi'$ (so $V \cong V'$), then*

$$\mathrm{Ext}^1_{\mathfrak{M}}(k_0, k_0)^{|\mathbf{x}|-1} \oplus \mathrm{Ext}^1_{\mathfrak{M}}(V, V) \cong \bigoplus_{x \in \mathbf{x}} \mathrm{Ext}^1_{\mathfrak{M}}(V_x, V'_x).$$

Conclusion: Reduced to computation of extensions at the same point.

Reductive Lie algebras

For any f.d. reductive Lie algebra L, we set

$$L_{\mathrm{ss}} = [L, L], \quad L_{\mathrm{ab}} = Z(L) \cong L/[L, L],$$

so $L = L_{\mathrm{ss}} \oplus L_{\mathrm{ab}}$.

Proposition (Modules for reductive Lie algebras)

Any small module for a f.d. reductive Lie algebra L is of the form

$$V_{\mathrm{ss}} \otimes V_{\mathrm{ab}}$$

where V_{ss} is a small L_{ss}-module, and V_{ab} is a small L_{ab}-module.

Lemma (Bourbaki)

Since $\mathfrak{g}$ is reductive, $\mathfrak{g}^x$ is reductive for all $x \in X$.

Extensions between evaluation modules at a single point

Fix a point $x \in X$ and define

$$\mathfrak{K} = \ker(\mathrm{ev}_x), \quad \mathfrak{Z} = \mathrm{ev}_x^{-1}(\mathfrak{g}^x_{\mathrm{ab}}) = \{\alpha \in \mathfrak{M} \mid [\alpha, \mathfrak{M}] \subseteq \mathfrak{K}\}.$$

Theorem (Neher-S. 2011)

Suppose V and V' are two evaluation modules at the point x. Then

$$\mathrm{Ext}^1_{\mathfrak{M}}(V, V') = \begin{cases} \mathrm{Hom}_{\mathfrak{g}^x}(\mathfrak{K}_{\mathrm{ab}}, V^* \otimes V') & \text{if } V_{\mathrm{ab}} \not\cong V'_{\mathrm{ab}}, \\ \mathrm{Hom}_{\mathfrak{g}^x_{\mathrm{ss}}}(\mathfrak{Z}_{\mathrm{ab}}, V^* \otimes V') & \text{if } V_{\mathrm{ab}} \cong V'_{\mathrm{ab}}. \end{cases}$$

Proposition (Neher-S. 2011, $\Gamma = \{1\}$ case due to Kodera)

If V, V' are evaluation modules at x, $\mathfrak{g}$ is semisimple, Γ is abelian, and Γ_x is trivial, then

$$\mathrm{Ext}^1_{\mathfrak{M}}(V, V') = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, V^* \otimes V') \otimes (I/I^2)^\Gamma,$$

where $I = \{f \in A \mid f(\Gamma \cdot x) = 0\}$.

Block decompositions

For an arbitrary Lie algebra L, let

$$\mathcal{F} = \text{category of f.d. reps.}$$

Then

- $\mathcal{F}$ is an abelian tensor category, and
- any object in $\mathcal{F}$ can be written uniquely as a sum of indecomposables.

$\mathcal{F}$ admits a unique decomposition into a direct sum of indecomposable abelian subcategories

$$\mathcal{F} = \bigoplus_{\beta} \mathcal{F}_\beta.$$

The subcategories $\mathcal{F}_\beta$ are the blocks of $\mathcal{F}$.

Block decompositions

Definition (Linked)

Suppose $U, V \in \mathcal{F}$ are indecomposable. We say U and V are linked if there exist indecomposable L-modules

$$U = U_1,\ U_2,\ \dots,\ U_n = V,$$

such that

$$\mathrm{Hom}_L(U_k, U_{k+1}) \neq 0 \quad \text{or} \quad \mathrm{Hom}_L(U_{k+1}, U_k) \neq 0 \quad \forall\ 1 \leq k < n.$$

We say that arbitrary $U, V \in \mathcal{F}$ are linked if every indecomposable summand of U is linked to every indecomposable summand of V.

Fact: The equivalences classes of linked objects are precisely the blocks of $\mathcal{F}$.

Block decompositions

For $x \in X$, define

$$\mathcal{F}_x = \text{category of eval reps with support } \Gamma \cdot x,$$
$$\mathcal{B}_x = \text{blocks of the category } \mathcal{F}_x.$$

For $\gamma \in \Gamma$, the categories $\mathcal{F}_x$ and $\mathcal{F}_{\gamma \cdot x}$ are the same.

So we can define an action of Γ on $\mathcal{B}_X = \bigsqcup_{x \in X} \mathcal{B}_x$ by letting

$$\gamma : \mathcal{B}_x \to \mathcal{B}_{\gamma \cdot x}, \quad \gamma \in \Gamma,$$

be the identification.

Block decompositions

Definition

Let $\mathfrak{B}_X$ be the set of finitely supported equivariant maps $X \to \mathcal{B}_X$ mapping x to $\mathcal{B}_x$ for all $x \in X$.

Definition

Let $\mathcal{F}_{\mathrm{eval}}$ be the full subcategory of $\mathcal{F}$ whose objects are those reps whose irreducible constituents are eval reps.

Remark: If all small reps are eval reps, then $\mathcal{F}_{\mathrm{eval}} = \mathcal{F}$.

Theorem (Neher-S. 2011)

The blocks of $\mathcal{F}_{\mathrm{eval}}$ are naturally parameterized by $\mathfrak{B}_X$.

For a more explicit description, one needs a more explicit description of

$$\mathcal{B}_x, \quad x \in X.$$

Application: untwisted map algebras

If $\Gamma = \{1\}$ and $\mathfrak{g}$ is semisimple, one can show that

$$\mathcal{B}_x \cong P/Q \quad \forall\, x \in X,$$

where

$$P = \text{weight lattice of } \mathfrak{g},$$
$$Q = \text{root lattice of } \mathfrak{g}.$$

We recover a result of Kodera.

Corollary (Neher-S. 2011)

Under the above assumptions, the blocks of the category of finite-dimensional (evaluation) modules are naturally enumerated by finitely-supported maps

$$X \to P/Q.$$

In the case of the untwisted loop algebra, this is due to Chari-Moura.

Application: free groups actions (multiloop algebras)

If Γ acts freely on X and $\mathfrak{g}$ is semisimple, then

$$\mathcal{B}_x \cong P/Q \quad \forall\, x \in X.$$

Multiloop algebras satisfy this condition.

Corollary (Neher-S. 2011)

Under the above assumptions, the blocks of the category of finite-dimensional (evaluation) modules are naturally enumerated by finitely-supported equivariant maps

$$X \to P/Q.$$

In the special case of the (single, twisted) loop algebra, this recovers a result of Senesi.

Application: generalized Onsager algebras

For a generalized Onsager algebra, we know that all small reps are eval reps.

Corollary (Neher-S. 2011)

The blocks of the category of f.d. modules of a generalized Onsager algebra are naturally parameterized by finitely-supported equivariant maps

$$X \to (P/Q) \sqcup (P_0/Q_0), \text{ such that}$$
$$X \setminus \{\pm 1\} \to P/Q, \ \{\pm 1\} \to P_0/Q_0,$$

where P_0, Q_0 are the weight and root lattices of $\mathfrak{g}^\Gamma$.

Local Weyl modules

Assume $\mathfrak{g}$ is semisimple and choose a set of Chevalley generators

$$\{e_i, f_i, h_i\}_{i \in I}.$$

This gives triangular decompositions

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$
$$(\mathfrak{g} \otimes A) = (\mathfrak{n}^- \otimes A) \oplus (\mathfrak{h} \otimes A) \oplus (\mathfrak{n}^+ \otimes A).$$

Definition (Local Weyl module)

For $\psi \in \mathcal{E}$, the (untwisted) local Weyl module $W(\psi)$ is the $(\mathfrak{g} \otimes A)$-module generated by a nonzero vector w_ψ satisfying

$$(\mathfrak{n}^+ \otimes A) \cdot w_\psi = 0, \quad (f_i \otimes 1)^{\lambda(h_i)+1} \cdot w_\psi = 0, \quad i \in I,$$
$$\alpha \cdot w_\psi = \left(\sum_{x \in \operatorname{supp} \psi} \psi(x)(\alpha(x)) \right) w_\psi, \quad \alpha \in \mathfrak{h} \otimes A,$$

where $\lambda = \sum_{x \in \mathbf{x}} \psi(x)$.

Twisting and untwisting functors

Assume Γ acts freely on X (so all small reps are eval reps).

We define the support of a finite-dimensional module to be the union of the supports of its irreducible constituents.

For a finite subset $\mathbf{x} \subseteq X$ which does not contain two points in the same orbit, we have twisting and untwisting functors:

$$\begin{array}{c}\text{Category of f.d. reps of } \mathfrak{g} \otimes A \text{ with support in } \mathbf{x} \\ \mathbf{U_x} \uparrow \quad \downarrow \mathbf{T_x} \\ \text{Category of f.d. reps of } (\mathfrak{g} \otimes A)^\Gamma \text{ with support in } \Gamma \cdot \mathbf{x}\end{array}$$

These are isomorphisms of categories.

Use: allows us to move back and forth between the twisted and untwisted settings.

Twisted local Weyl modules: Definition

Definition (Twisted local Weyl module (Fourier-Khandai-Kus-S. '11))

For $\psi \in \mathcal{E}$, let $\mathbf{x} \subseteq \operatorname{supp} \psi$ contains one point in each Γ-orbit. Then define the twisted local Weyl module

$$W_\Gamma(\psi) = \mathbf{T}_x(W(\psi \cdot 1_{\mathbf{x}})),$$

where $1_{\mathbf{x}}$ is the characteristic function of $\mathbf{x}$, and $W(\psi \cdot 1_{\mathbf{x}})$ is the usual (untwisted) Weyl module for $\mathfrak{g} \otimes A$.

Note: The definition is independent of the choice of $\mathbf{x} \subseteq \operatorname{supp} \psi$ (up to isom).

In the case of twisted loop algebras, this coincides with a definition given by Chari-Fourier-Senesi.

Twisted local Weyl modules: properties

The twisted local Weyl module have properties analogous to the usual (untwisted) Weyl modules.

1. $W_\Gamma(\psi)$ has a unique irreducible quotient corresponding to ψ.
2. Every "maximal weight module" of "maximal weight" ψ is a quotient of $W_\Gamma(\psi)$.
3. The $W_\Gamma(\psi)$ have a characterization in terms of homological properties (a twisted analogue of a characterization given by Chari-Fourier-Khandai in the untwisted case).
4. The local Weyl modules have a natural tensor product property:

$$W_\Gamma(\psi + \varphi) \cong W_\Gamma(\psi) \otimes W_\Gamma(\varphi)$$

   if $\operatorname{supp}\psi \cap \operatorname{supp}\varphi = \varnothing$.

Further directions

Can one describe the finite-dimensional representations (not necessarily irreducible)?

Weyl modules:

- Case where Γ is not abelian or does not act freely?
- Global Weyl modules?
- Weyl functor?

Higher Ext groups?