Crystals, quiver varieties and coboundary categories for Kac-Moody algebras

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Symmetric group

Definition - Symmetric group

The symmetric group S_n

- generators $s_1, \ldots, s_{n-1}$
- relations
  1. $s_i s_j = s_j s_i$ for $|i-j| \geq 2$
  2. Yang-Baxter equation:
  $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i \leq n-2$$
  3. $s_i^2 = 1$ for $1 \leq i \leq n-1$

Symmetric group as permutations

$$S_n = \text{group of permutations of } \{1, 2, \ldots, n\}$$
$$s_i = (i, i+1)$$

Braid group

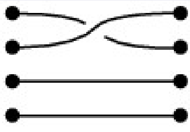
Definition - Braid group

The *n*-strand Braid group $\mathcal{B}_n$ $(n \geq 1)$

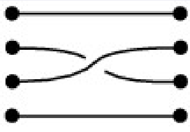
- generators $\sigma_1, \ldots, \sigma_{n-1}$
- braid relations:
  1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$
  2. Yang-Baxter equation:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-2$$

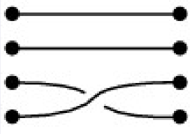
Example: $\mathcal{B}_4$



σ_1



σ_2



σ_3

Cactus group

For $1 \leq p < q \leq n$, let

$$\hat{s}_{p,q} = \begin{pmatrix} 1 & \cdots & p-1 & p & p+1 & \cdots & q & q+1 & \cdots & n \\ 1 & \cdots & p-1 & q & q-1 & \cdots & p & q+1 & \cdots & n \end{pmatrix} \in S_n.$$

Since $\hat{s}_{i,i+1} = s_i$, these elements generate S_n.

If $1 \leq p < q \leq n$ and $1 \leq k < l \leq n$, we say

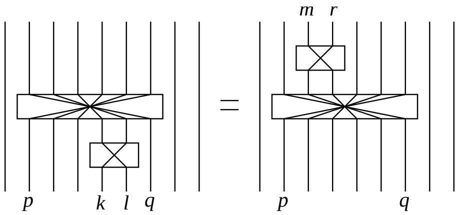
- $p < q$ and $k < l$ are disjoint if $q < k$ or $l < p$
- $p < q$ contains $k < l$ if $p \leq k < l \leq q$

Cactus group

Definition – Cactus group

The *n-fruit cactus group* J_n

- generators $s_{p,q}$ for $1 \leq p < q \leq n$
- relations
  1. $s_{p,q}^2 = 1$,
  2. $s_{p,q} s_{k,l} = s_{k,l} s_{p,q}$ if $p < q$ and $k < l$ are disjoint, and
  3. $s_{p,q} s_{k,l} = s_{m,r} s_{p,q}$ if $p < q$ contains $k < l$, where $m = \hat{s}_{p,q}(l)$ and $r = \hat{s}_{p,q}(k)$.



Monoidal categories

Definition - Monoidal category

- category $\mathcal{C}$
- functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- associator $\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$ such that

$$
\begin{array}{ccccc}
 & & ((U \otimes V) \otimes W) \otimes X & & \\
 & \swarrow^{\alpha_{U,V,W} \otimes \mathrm{id}_X} & & \searrow^{\alpha_{U \otimes V, W, X}} & \\
(U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\
\downarrow \alpha_{U, V \otimes W, X} & & & & \downarrow \alpha_{U,V,W \otimes X} \\
U \otimes ((V \otimes W) \otimes X) & & \xrightarrow{\mathrm{id}_U \otimes \alpha_{V,W,X}} & & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

commutes (pentagon axiom)

Braided monoidal categories

Definition - Braided monoidal category

- monoidal category $\mathcal{C}$
- natural isoms $\sigma^{br}_{U,V} : U \otimes V \to V \otimes U$ (braiding)
- for all $U, V, W \in \mathrm{Ob}\,\mathcal{C}$

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{\mathrm{Id} \otimes \sigma^{br}_{V,W}} & U \otimes W \otimes V \\ & \searrow^{\sigma^{br}_{U \otimes V, W}} & \downarrow{\scriptstyle \sigma^{br}_{U,W} \otimes \mathrm{Id}} \\ & & W \otimes U \otimes V \end{array}$$

commutes

Definition - Symmetric monoidal category

- braided monoidal category $\mathcal{C}$
- $\sigma^{br}_{V,U} \circ \sigma^{br}_{U,V} = \mathrm{id}_{U \otimes V}$ for all $U, V \in \mathrm{Ob}\,\mathcal{C}$

Coboundary monoidal (cactus) categories

Definition - Coboundary monoidal category [Drinfeld '90]

- monoidal category $\mathcal{C}$
- natural isoms $\sigma_{U,V} : U \otimes V \to V \otimes U$ (commutor)
- symmetry axiom: $\sigma_{V,U} \circ \sigma_{U,V} = 1$
- cactus relation: for all $U, V, W \in \operatorname{Ob} \mathcal{C}$,

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{\operatorname{Id} \otimes \sigma_{V,W}} & U \otimes W \otimes V \\ \Big\downarrow{\scriptstyle \sigma_{U,V} \otimes \operatorname{Id}} & & \Big\downarrow{\scriptstyle \sigma_{U,W \otimes V}} \\ V \otimes U \otimes W & \xrightarrow{\sigma_{V \otimes U, W}} & W \otimes V \otimes U \end{array}$$

  commutes

Group actions on categories

Group actions

Consider action on n-fold tensor product generated by action of σ^{br} or σ on nearest neighbors

Symmetric monoidal category: yields action of symmetric group S_n

Braided monoidal category: yields action of braid group B_n

Coboundary category: yields action of cactus group J_n (can build $s_{p,q}$'s out of σ)

Braid and cactus groups - geometric interpretations

Braid group and the punctured disc

$\mathcal{B}_n \cong$ fundamental group of configuration space of n points in the unit disc

$$\mathcal{B}_n \twoheadrightarrow S_n \quad \sigma_i \mapsto s_i$$

Kernel is the pure braid group (marked points)

Cactus group and moduli space of curves

Surjective group homomorphism

$$J_n \twoheadrightarrow S_n \quad s_{p,q} \mapsto \hat{s}_{p,q}$$

Kernel is fundamental group of Deligne-Mumford compactification of moduli space of real genus zero curves with $n+1$ marked points

Elements of moduli space look like cacti from genus *Opuntia*

Braid group vs. cactus group

Question: What is the precise relationship between the braid group and the cactus group?

Neither is a quotient of the other

Cactus group injects into a completion of the braid group (more on this later)

Quantum groups

Notation

- $\mathfrak{g}$ - Kac-Moody algebra
- $U(\mathfrak{g})$ - Universal enveloping algebra
- $U_q(\mathfrak{g})$ - Quantum group

Representation theory

Tensor product in $U(\mathfrak{g})$ is symmetric:

$$U \otimes V \cong V \otimes U, \quad u \otimes v \mapsto v \otimes u$$

Reps of $U(\mathfrak{g})$ form a symmetric monoidal category

$U_q(\mathfrak{g})$ is a q-deformation of $U(\mathfrak{g})$

Reps of $U(\mathfrak{g})$ can be q-deformed to yield reps of $U_q(\mathfrak{g})$.

Quantum groups - the R-matrix

Tensor product in $U_q(\mathfrak{g})$ is not symmetric:

flip : $u \otimes v \mapsto v \otimes u$ is not a homomorphism

The R-matrix

For $\mathfrak{g}$ of finite type, recall the R-matrix,

$$R \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$$

satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and so yields a braiding

$$\sigma^{br} = \text{flip} \circ R : U \otimes V \to V \otimes U$$

Quantum groups - a commutor

Unitarized R-matrix [Drinfeld '90]

Define

$$R' = R(\mathrm{flip}(R) \cdot R)^{-1/2}$$
$$\sigma' = \mathrm{flip} \circ R'$$

Then σ' satisfies the cactus relation

Theorem

Category of reps of $U_q(\mathfrak{g})$ is both a braided and a coboundary monoidal category ($\mathfrak{g}$ of finite type)

R-matrix vs. commutor

R-matrix

- gives rise to a braiding
- yields braid group action on multiple tensor products
- makes category of $U_q(\mathfrak{g})$-modules a braided monoidal category

Commutor

- yields cactus group action
- makes categories of $U_q(\mathfrak{g})$-modules a coboundary monoidal (or cactus) category

Crystals

Crystals – combinatorial representation theory

- crystals are $q \to 0$ limit of theory of quantum groups
- representation theory $\longrightarrow$ combinatorics

$$\text{Reps of } U(\mathfrak{g}) \xleftarrow{q \to 1} \text{Reps of } U_q(\mathfrak{g}) \xrightarrow{q \to 0} \text{Crystals}$$

Rep of $U_q(\mathfrak{g})$	**Crystal**
Basis	Vertices (with weight)
Action of f_i	Colored directed edge

Crystals

Uses of crystals

- rep theory replaced by easier combinatorial models
- combinatorial models contain info about
  - weights (characters)
  - decomposition into irreducibles
  - tensor product decompositions

Crystal tensor product

Crystal tensor product

B_1, B_2 - crystals

$$B_1 \otimes B_2 = B_1 \times B_2 \text{ as sets}$$

Purely combinatorial formula for action of crystal operators (coloured edges) in terms of crystals B_1, B_2

Category of crystals is a monoidal category

(Lack of) Symmetry

Product is not symmetric

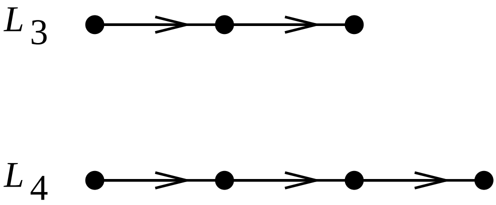
$$\text{i.e. } (b_1, b_2) \mapsto (b_2, b_1) \text{ is not a crystal map}$$

Crystal tensor product - example

$\mathfrak{g} = \mathfrak{sl}_2$

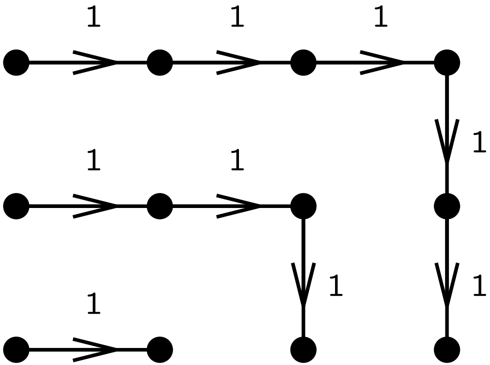
Consider $L_3 \otimes L_4$

Crystal graphs are



Crystal tensor product - example

Tensor product rule yields



Thus

$$L_3 \otimes L_4 = L_2 \oplus L_4 \oplus L_6$$

Category of crystals

Recall: category of reps of $U_q(\mathfrak{g})$ is a braided monoidal category (via R-matrix)

"Problem": R-matrix does not pass to the $q \to 0$ limit.

Proposition

The category of crystals cannot be given the structure of a braided monoidal category (even for $\mathfrak{sl}_2$!!)

Question: Can the category of crystals be given the structure of a coboundary category?

Crystal commutor - Henriques-Kamnitzer (Berenstein) '04

Notation

- $\mathfrak{g}$ - complex simple Lie algebra
- I - set of vertices of Dynkin diagram
- w_0 - long element in Weyl group
- $\theta : I \to I$, involution s.t. $-w_0 \cdot \alpha_i = \alpha_{\theta(i)}$
- B_λ - crystal corresponding to $\mathfrak{g}$-mod of h.w. λ

Schützenberger involution ξ

ξ - (set) involution on B_λ

- acts by w_0 on weights
- interchanges action of e_i and $f_{\theta(i)}$

Crystal commutor - Henriques-Kamnitzer (Berenstein) '04

Definition

Commutor: Morphism of crystals

$$\begin{aligned} \sigma_{A,B} : A \otimes B &\longrightarrow B \otimes A \\ a \otimes b &\longmapsto \xi(\xi(b) \otimes \xi(a)) \end{aligned}$$

Properties

1. $\sigma_{B,A} \circ \sigma_{B,A} = 1$
2. Cactus Relation: Commutative diagram

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{1 \otimes \sigma_{B,C}} & A \otimes C \otimes B \\ {\scriptstyle \sigma_{A,B} \otimes 1} \downarrow & & \downarrow {\scriptstyle \sigma_{A,C \otimes B}} \\ B \otimes A \otimes C & \xrightarrow{\sigma_{B \otimes A, C}} & C \otimes B \otimes A \end{array}$$

Crystal commutor

Theorem [Henriques-Kamnitzer '04]

For $\mathfrak{g}$ of finite type, the category of crystals is a coboundary category (via the commutor σ)

Schützenberger involution ξ defined using long element of Weyl group

Thus, ξ (and hence σ) not defined for infinite type

Goal: Generalize commutor to symmetrizable Kac-Moody algebras

Kashiwara involution

B_∞ - crystal corresponding to $U_q^-(\mathfrak{g})$

Definition – Kashiwara Involution

Let $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the anti-automorphism given by

$$q^* = q, \quad e_i^* = e_i, \quad f_i^* = f_i, \quad q(h)^* = q(-h).$$

The map $*$ is called Kashiwara's involution

- sends $U_q^-(\mathfrak{g})$ to $U_q^-(\mathfrak{g})$
- induces a map $* : B_\infty \to B_\infty$
- preserves weights

Crystals B_λ "sit inside" B_∞:

$$b \in B_\lambda \text{``}\subset\text{''} B_\infty,$$
$$*b \in B_\mu \quad \text{for certain } \mu$$

Crystal commutor - alternate definition

All highest weight elements of $B_\lambda \otimes B_\mu$ are of the form

$$b_\lambda \otimes b$$

for $b \in B_\mu$

Theorem [Kamnitzer-Tingley '06]

If $b_\lambda \otimes b$ is highest weight in $B_\lambda \otimes B_\mu$ then

$$\sigma_{B_\lambda, B_\mu}(b_\lambda \otimes b) = b_\mu \otimes *b$$

Note: Commutor determined by action on highest weight elements

We can use this definition to extend σ to arbitrary symmetrizable Kac-Moody algebras

Crystal commutor - two definitions

Definition 1 – Using Schützenberger involution

- only defined for finite type
- easy to see cactus relation satisfied

Definition 2 – Using Kashiwara involution

- defined for arbitrary type
- not known if cactus relation satisfied (for infinite type)

Note: Drinfeld's unitarized R-matrix (in the $q \to 0$ limit) corresponds with the crystal commutor [Kamnitzer-Tingley '07]

Quiver varieties - overview

Quiver = directed graph

Idea: Associate varieties X to reps of $U(\mathfrak{g})$ such that

$$\text{rep} \longleftrightarrow H_*(X)$$
$$\text{crystal} \longleftrightarrow \{\text{Irred comps of } X\}$$

$$U^-(\mathfrak{g}) \longleftrightarrow \text{Lusztig quiver varieties}$$
$$\text{Irred integrable h.w. reps} \longleftrightarrow \text{Nakajima quiver varieties}$$

Lusztig quiver varieties

$\mathfrak{g}$ - symmetric Kac-Moody algebra
$I = \{\text{vertices in Dynkin diagram}\}$
$Q =$ double quiver (directed graph) on Dynkin diagram
$H =$ set of directed edges ($h \mapsto \overline{h}$ reverse orientation)

$$V = \bigoplus_{i \in I} V_i, \quad v = (\dim V_i)_{i \in I}$$

Definition

A representation of Q is a collection

$$x = (x_h)_{h \in H}, \quad x_h : V_{\mathrm{tail}(h)} \to V_{\mathrm{tip}(h)}$$

Example: $\mathfrak{sl}_{n+1}$

Consider the Dynkin graph of $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Take *both* orientations of each edge to get Q

Reps of Q are elements of

$$\bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{tail}(h)}, V_{\mathrm{tip}(h)})$$

$$= \bigoplus_{i=1}^{n-1} \mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i)$$

Lusztig quiver varieties

Definition - Lusztig quiver variety [Lusztig '90]

$\Lambda(v)$ = set of quiver reps x such that

- x is nilpotent
- for all $i \in I$

$$\sum_{h:\mathrm{tail}(h)=i} (\pm) x_{\bar{h}} x_h = 0$$

Lusztig quiver varieties

Geometric realization of $U^-(\mathfrak{g})$ [Lusztig '90]

$$\bigoplus_v H_{\text{top}}(\Lambda(v)) \cong U^-(\mathfrak{g})$$

$$H_{\text{top}}(\Lambda(v)) \cong -\sum_i v_i \alpha_i \text{ weight space}$$

α_i = simple roots

Geometric realization of B_∞ [Kashiwara-Saito '97]

$$\left\{ \text{Irred comps of } \bigsqcup_v \Lambda(v) \right\} \cong B_\infty \quad \text{(as crystals)}$$

$$X_b \leftrightarrow b$$

Nakajima quiver varieties

$\mathfrak{g}$ - symmetric Kac-Moody algebra
$I = \{\text{vertices in Dynkin diagram}\}$
$Q = \text{double quiver (directed graph) on Dynkin diagram}$

$$V = \bigoplus_{i \in I} V_i, \quad v = (\dim V_i)_{i \in I}$$

$$W = \bigoplus_{i \in I} W_i, \quad w = (\dim W_i)_{i \in I}$$

$$M(v, w) = \bigoplus_{i \to j \in Q} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, W_i) \oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i)$$

Nakajima quiver varieties

Definition [Nakajima '94, '98]

Quiver variety: Hyper-Kähler quotient

$$\mathfrak{M}(v, w) = M(v, w)///_{\zeta} \prod_i GL(V_i)$$

Lagrangian subvariety:

$$\mathfrak{L}(v, w) \subset \mathfrak{M}(v, w)$$

$$\mathfrak{L}(w) = \bigsqcup_v \mathfrak{L}(v, w)$$

Nakajima quiver varieties

Geometric realization of Irred hw $\mathfrak{g}$-mods

$$\bigoplus_v H_{\mathrm{top}}(\mathfrak{L}(v,w)) \cong L_\lambda, \quad \lambda = \sum_i w_i \omega_i$$

$$H_{\mathrm{top}}(\mathfrak{L}(v,w)) \cong L_\lambda^\mu, \quad \mu = \lambda - \sum_i v_i \alpha_i$$

ω_i = fundamental weights
α_i = simple roots

Nakajima quiver varieties

Geometric realization of crystal graph [Saito '02]

Vertex Set: Irr comps of $\mathfrak{L}(w)$

Crystal Operators: Natural geometric definition. $\tilde{f}_i$ increases dim of V_i by one

$$\{\text{Irr comp } \mathfrak{L}(w)\} \cong B_\lambda \quad \text{(as crystals)}$$

$\lambda = \sum_i w_i \omega_i$

Irreducibles from Verma modules

Algebraic construction

$\mathbb{C}_\lambda$ - $\mathfrak{h}$ acts by λ

Verma module: $M_\lambda = U^-(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ "shift weights"

Irreducible: $L_\lambda = M_\lambda / I_\lambda$ (I_λ maximal submodule) "cut"

Geometric construction

$$U^-(\mathfrak{g}) \longleftrightarrow \text{Lusztig quiver variety}$$

$$\text{"shift weights"} \longleftrightarrow \text{add in } W_i\text{'s}$$

$$\text{"cut"} \longleftrightarrow \text{quotient by } \prod_i GL(V_i)$$

Tensor product quiver varieties

Goal: Give geometric realization of tensor product

$$L_{\lambda_1} \otimes \cdots \otimes L_{\lambda_n}$$

Pick decomposition

$$W = W^1 \oplus W^2 \oplus \cdots \oplus W^n$$

Action of $\mathbb{C}^*$ on W by

$$\lambda(t) = 1_{W^1} \oplus t1_{W^2} \oplus \cdots \oplus t^{n-1}1_{W^n}$$

Induces action on $\mathfrak{M}(w)$

Tensor product quiver variety

$$\mathfrak{T}(w^1, \ldots, w^n) = \left\{ x \in \mathfrak{M}(w) \;\middle|\; \lim_{t \to 0} \lambda(t) \cdot x \in \mathfrak{L}(w^1) \times \cdots \times \mathfrak{L}(w^n) \right\}$$

Note: Def is not symmetric

Tensor product quiver varieties

Geometric realization of tensor product [Malkin, Nakajima '01]

$$H_{\text{top}}(\mathfrak{T}(w^1, \dots, w^n)) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

as $\mathfrak{g}$-modules.

$$\{\text{Irr comp } \mathfrak{T}(w^1, \dots, w^n)\} \cong B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}, \quad Y_b \leftrightarrow b$$

as crystals

Geometric Kashiwara involution

Suppose

$$x = (x_h)_{h \in H} \in \bigoplus_{i \to j \in Q} \mathrm{Hom}(V_i, V_j)$$

Fix hermitian form on V. Then

$$x \mapsto x^\dagger, \quad (x^\dagger)_h \stackrel{\text{def}}{=} x_{\bar{h}}^\dagger$$

induces involution of $\Lambda(v)$

Proposition [Kashiwara-Saito '97]

Above map gives geometric realization of Kashiwara involution:

$$X_b^\dagger = X_{*b}, \quad b \in B_\infty$$

Geometric commutor

Suppose

$$(x, s, t) \in M(v, w) = \bigoplus_{i \to j \in Q} \operatorname{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i) \oplus \bigoplus_{i \in I} \operatorname{Hom}(W_i, V_i)$$

Fix hermitian forms on V, W. Then

$$(x, s, t)^\dagger \stackrel{\text{def}}{=} (x^\dagger, t^\dagger, s^\dagger) \in M(v, w)$$

Map

$$(x, s, t) \mapsto (x, s, t)^\dagger$$

induces involution of $M(v, w)$

Geometric commutor

Proposition

For (x, s, t) "highest weight",

$$(x, s, t) \in \mathfrak{T}(w^1, \dots, w^n) \iff (x, s, t)^\dagger \in \mathfrak{T}(w^n, \dots, w^1)$$

Recall: Commutor determined by action on highest weight elements

Theorem [S '08]

Above map gives the crystal commutor: for $b_1 \otimes b_2$ highest weight

$$Y^\dagger_{b_1 \otimes b_2} = Y_{\sigma(b_1 \otimes b_2)}$$

The cactus relation

Cactus relation: for all $U, V, W \in \operatorname{Ob}\mathcal{C}$,

$$\begin{array}{ccc} U \otimes V \otimes W & \xrightarrow{\operatorname{Id} \otimes \sigma_{V,W}} & U \otimes W \otimes V \\ {\scriptstyle \sigma_{U,V} \otimes \operatorname{Id}} \downarrow & & \downarrow {\scriptstyle \sigma_{U, W \otimes V}} \\ V \otimes U \otimes W & \xrightarrow{\sigma_{V \otimes U, W}} & W \otimes V \otimes U \end{array}$$

commutes

Proposition [S '08]

For $b = b_1 \otimes b_2 \otimes b_3 \in B(\lambda_1) \otimes B(\lambda_2) \otimes B(\lambda_3)$,

$$Y_{\sigma_{U,W \otimes V} \circ (\operatorname{Id} \otimes \sigma_{V,W})(b)} = Y_b^\dagger = Y_{\sigma_{V \otimes U, W} \circ (\sigma_{U,V} \otimes \operatorname{Id})(b)}$$

Symmetric Kac-Moody algebras and coboundary categories

Theorem [S '08]

$\mathfrak{g}$ - Kac-Moody algebra with symmetric Cartan matrix

Category of crystals is a coboundary monoidal category

Question: What about symmetrizable Kac-Moody algebras (i.e. remove condition that Cartan matrix symmetric)?

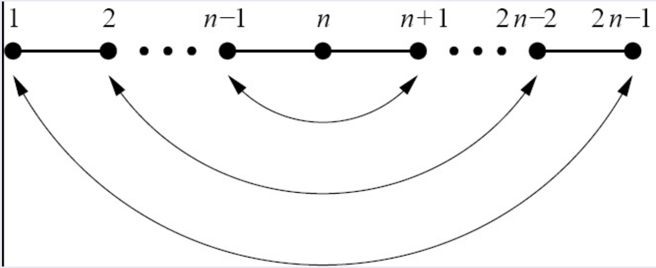
Answer: Can use "folding" procedure to get general symmetrizable case from symmetric case.

Folding Dynkin diagrams

Well-known procedure for producing non-simply laced Dynkin diagrams from simply-laced ones via "folding"

Example: Type B_n ($\mathfrak{g} = \mathfrak{so}_{2n+1}$)

Realized via folding of type A_{2n-1} diagram



Folding

Folding quiver varieties [S '04]

- folding induces automorphism of quiver varieties
- fixed irred comps yield geometric realization for non-simply laced types

Folding crystals

Crystals for non-simply laced type sit inside crystals for simply-laced type

Symmetrizable Kac-Moody algebras and coboundary categories

Theorem [S '08]

$\mathfrak{g}$ - symmetrizable Kac-Moody algebra

Category of crystals is a coboundary monoidal category

Recall: Category of crystals is not a braided monoidal category