Quivers and the Euclidean Group

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Euclidean group

Definition (Euclidean group)

Group of orientation-preserving isometries of n-dim Euclidean space:

$$E(n) = \mathbb{R}^n \rtimes SO(n)$$

Study (at least for $n = 2, 3$) predates even concept of group.

We will focus on $E(2)$ – much still unknown about rep theory.

Representations of the Euclidean group

- $E(2)$ solvable $\Rightarrow$ all finite-dim irreps are 1-dim
- finite-dim unitary reps (of interest in quantum mechanics) are completely reducible $\Rightarrow$ isom to direct sum of one-dim reps
- infinite-dim unitary reps have received considerable attention
- $\exists$ finite-dim nonunitary indecomp reps (not irreducible)
    - much less known about these
    - play important role in math physics and rep theory of Poincaré group

A little mathematical physics

Poincaré Group

Group of isometries of Minkowski spacetime

$$\text{Poincaré group} = \{\text{translations}\} \rtimes \text{Lorentz group}$$

The Little Group (Wigner 1939)

Def: maximal subgroup of Lorentz group leaving invariant the four-momentum of a particle

- governs internal space-time symmetries of particle
- massive particles: little group locally isom to $O(3)$
- massless particles: little group locally isom to $E(2)$

A bit more mathematical physics

Gravity

Consider

- Chern-Simons formulation of Einstein gravity
- $2 + 1$ dimensions
- space-time with Euclidean signature
- vanishing cosmological constant

Then phase space of gravity is moduli space of flat $E(2)$-connections

The Euclidean algebra

Recall $E(2) = \mathbb{R}^2 \rtimes SO(2)$

The Euclidean Algebra

$\mathfrak{e}(2)$ = complexification of Lie alg of $E(2)$

Has basis $\{p_+, p_-, l\}$ and relations

$$[p_+, p_-] = 0, \quad [l, p_\pm] = \pm p_\pm$$

Representation Theory

$SO(2)$ compact $\Rightarrow$ finite-dim $E(2)$-modules equiv to finite-dim $\mathfrak{e}(2)$-modules where l acts semisimply with integer eigenvalues

We use term $\mathfrak{e}(2)$-module to mean such a module

Weight decompositions

V an $\mathfrak{e}(2)$-module

We have weight decomposition into I-eigenspaces

$$V = \bigoplus_{k \in \mathbb{Z}} V_k, \quad V_k = \{v \in V \mid I \cdot v = kv\}$$

and

$$p_+ V_k \subseteq V_{k+1}, \quad p_- V_k \subseteq V_{k-1}$$

We define

$$\mathbf{dim}\, V = (\dim V_k)_{k \in \mathbb{Z}} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$$

Modified enveloping algebra

U = universal enveloping algebra of $\mathfrak{e}(2)$

U^+, U^-, U^0 subalgebras generated by p_+, p_-, l

Have triangular decomp $U \cong U^+ \otimes U^0 \otimes U^-$

Modified enveloping algebra

Following idea of Lusztig, define modified enveloping algebra

$$\tilde{U} = U^+ \otimes \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^-$$

with multiplication

$$a_k a_l = \delta_{kl} a_k$$

$$p_+ a_k = a_{k+1} p_+, \quad p_- a_k = a_{k-1} p_-,$$

$$p_+ p_- a_k = p_- p_+ a_k$$

Representation theory

$$\tilde{U} = U^+ \otimes \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^-$$

$a_k \sim$ projection to kth weight space

Definition

A $\tilde{U}$-module is unital if

1. $\forall v \in V$, $a_k v = 0$ for almost all $k \in \mathbb{Z}$
2. $\forall v \in V$, $\sum_{k \in Z} a_k v = v$

$\tilde{U}$-module $\sim$ U-module with weight decomp

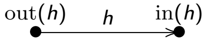
Proposition

$$\textbf{Mod}\, \tilde{U} \cong \textbf{Mod}\, U \cong \textbf{Mod}\, \mathfrak{e}(2)$$

Quivers

quiver = directed graph

$Q = (I, H)$
I = vertex set
H = (directed) edge set



Representations of quivers

- I-graded vector space $V = (V_i)_{i \in I}$
- linear map $x_h : V_{\mathrm{out}(h) \to \mathrm{in}(h)}$ for each $h \in H$

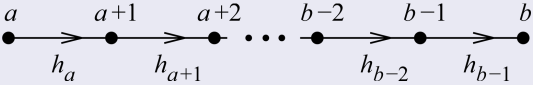
$$\mathrm{rep}(Q, V) = \bigoplus_{h \in H} Hom_{\mathbb{C}}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)})$$

Quivers

The quiver $Q_{a,b}$

$$I = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$$

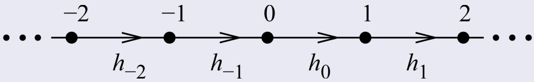
$$H = \{h_i \mid a \leq i \leq b-1\}, \quad \mathrm{out}(h_i) = i, \ \mathrm{in}(h_i) = i+1$$



The quiver Q_∞

$$I = \mathbb{Z}$$

$$H = \{h_i \mid i \in \mathbb{Z}\}, \quad \mathrm{out}(h_i) = i, \ \mathrm{in}(h_i) = i+1$$



Path algebra and double quiver

Path algebra

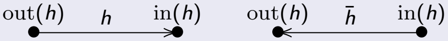
$\mathbb{C}Q$ = algebra spanned by paths with multiplication given by concatenation

$$\text{cat of reps of } Q \cong \mathbf{Mod}\,\mathbb{C}Q$$

Double quiver

Q^* = double quiver of Q

$$I_{Q^*} = I_Q,$$
$$H_{Q^*} = H_Q \cup \bar{H}_Q, \quad \bar{H}_Q = \{\bar{h} \mid h \in H_Q\}$$



Preprojective algebra

For $i \in I$, define

$$r_i = \sum_{h \in H,\, \mathrm{out}(h)=i} \bar{h}h - \sum_{h \in H,\, \mathrm{in}(h)=i} h\bar{h}$$

Preprojective algebra

$$P(Q) = \mathbb{C}Q^*/J$$

$J =$ two-sided ideal generated by r_i, $i \in I$

Representations of the preprojective algebra

$\mathrm{mod}(P(Q), V) = \{P(Q)\text{-modules with underlying v.s. } V\}$
Equivalent to set of elements of $\mathrm{rep}(Q^*, V)$ such that

$$\sum_{h \in H,\, \mathrm{out}(h)=i} x_{\bar{h}} x_h - \sum_{h \in H,\, \mathrm{in}(h)=i} x_h x_{\bar{h}} = 0 \quad \forall i \in I$$

Representation theory of the preprojective algebra

Proposition (Crawley-Boevey, Lusztig and others)

The following are equivalent for a quiver Q:

1. $P(Q)$ is finite-dimensional
2. All elements of $\mathrm{rep}(P(Q), V)$ are nilpotent
3. Q is a Dynkin quiver (underlying graph of ADE type)

Proposition

If Q is a finite quiver then

1. $P(Q)$ is of finite rep type iff Q is of Dynkin type A_n, $n \leq 4$
2. $P(Q)$ is of tame rep type iff Q is of Dynkin type A_5 or D_4
3. $P(Q)$ is of wild rep type for other types

Representation theory of $P(Q_{a,b})$ and $P(Q_\infty)$

Corollary

- $Q_{a,b}$ has finite rep type iff $b - a \leq 3$, and all reps are nilpotent
- Q_∞ is of wild rep type and all reps are nilpotent

Preprojective algebras and the Euclidean algebra

Theorem

The map $\psi : \mathbb{C}Q_\infty^* \to \tilde{U}$ given by (ϵ_i = trivial path at i)

$$\psi(\epsilon_i) = a_i, \quad \psi(h_i) = p_+ a_i = a_{i+1} p_+, \quad \psi(\bar{h}_i) = a_i p_- = p_- a_{i+1}$$

extends to a surjective map of algebras with kernel J. Thus

$$P(Q_\infty) \cong \tilde{U}$$

Corollary

$$\mathbf{Mod}\, \mathfrak{e}(2) \cong \mathbf{Mod}\, P(Q_\infty)$$

and

$$\mathbf{Mod}_{a,b}\, \mathfrak{e}(2) \cong \mathbf{Mod}\, P(Q_{a,b})$$

where $\mathbf{Mod}_{a,b}\, \mathfrak{e}(2)$ is category of $\mathfrak{e}(2)$-modules with weights lying between a and b

Representation theory of the Euclidean algebra

Theorem

- $\mathfrak{e}(2)$ (and hence $E(2)$) has wild representation type
- for $0 \leq b - a \leq 3$, $\exists$ a finite number of isom classes of indecomposable $\mathfrak{e}(2)$-modules whose weights lie between a and b

Lusztig quiver variety

Definition (Lusztig quiver variety)

$\Lambda_{V,Q}$ is set of all nilpotent $(x_h) \in \mathrm{mod}(P(Q), V)$

Recall, for Q of Dynkin type

$$\mathrm{mod}(P(Q), V) = \Lambda_{V,Q}$$

Relation to Kac-Moody algebras

Let $\mathfrak{g}_Q =$ Kac-Moody algebra whose Dynkin graph is underlying graph of Q.

$\#$ irred comps of $\Lambda_{V,Q} =$ dim of $(-\sum(\dim V_i)\alpha_i)$-weight space of $U(\mathfrak{g}_Q)^-$

Nakajima quiver variety

Let Q be Q_∞ or $Q_{a,b}$.

For I-graded vec spaces V and W, define

$$L_Q(V,W) = \Lambda_{V,Q} \oplus \bigoplus_{i \in I} \operatorname{Hom}_{\mathbb{C}}(W_i, V_i)$$

For $(x,s) = ((x_h)_{h \in H}, (s_i)_{i \in I})$ we say

- I-graded $S \subseteq V$ is x-invariant if

$$x_h(S_{\mathrm{out}(h)}) \subseteq S_{\mathrm{in}(h)} \ \forall h \in H$$

- (x,s) is stable if $\nexists$ proper x-invariant subspace of V containing $\operatorname{im} s$

Let $L_Q(V,W)^{\mathrm{st}}$ = set of stable points

Nakajima quiver variety

$G_V = \prod_{i \in I} GL(V_i)$ acts on $L_Q(V, W)$ by

$$g \cdot (x, s) = ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}), (g_i s_i))$$

Stabilizer in G_V of a stable point is trivial

Definition (Nakajima quiver variety)

$$\mathcal{L}_Q(V, W) = L_Q(V, W)^{\mathrm{st}} / G_V$$

Nakajima quiver varieties and Kac-Moody algebras

$$\bigoplus_V H_{\text{top}}(\mathcal{L}_Q(V,W)) \cong \text{irrep of } \mathfrak{g}_Q \text{ of hw } \sum_{i\in I} (\dim W_i)\omega_i$$

where ω_i are fundamental weights

$$H_{\text{top}}(\mathcal{L}_Q(V,W)) = \sum_{i\in I} (\dim W_i)\omega_i - \sum_{i\in I} (\dim V_i)\alpha_i \text{ weight space}$$

$$\# \text{ irred comps of } \mathcal{L}_Q(V,W) = \text{dim of weight space}$$

Representation theory of $\mathfrak{e}(2)$

- recall $\mathfrak{e}(2)$ has wild rep type
- restrict attention to subclasses of modules and attempt a classification
- impose restriction on number of generators of a module
- moduli spaces of such modules related to Nakajima quiver varieties

Moduli spaces of representations of $\mathfrak{e}(2)$

Let V be a rep of $\mathfrak{e}(2)$

We say $\{u_1, \ldots, u_n\} \subseteq V$ is a set of generators of V if

1. each u_i is a weight vector
2. $\nexists$ proper submodule of V containing all u_i

Definition

For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, let $E(\mathbf{v}, \mathbf{w})$ be set of all

$$(V, (u_k^j)_{k \in \mathbb{Z},\, 1 \leq j \leq \mathbf{w}_k})$$

where

- V is an $\mathfrak{e}(2)$-module with $\mathbf{dim}\, V = \mathbf{v}$
- $(u_k^j)_{k \in \mathbb{Z},\, 1 \leq j \leq \mathbf{w}_k}$ is a set of generators of V with wt $u_k^j = k$

Moduli spaces of representations of $\mathfrak{e}(2)$

Definition

We say

$$(V, (u_k^j)) \sim (\tilde{V}, (\tilde{u}_k^j))$$

if $\exists$ $\mathfrak{e}(2)$-module isom

$$\phi : V \xrightarrow{\cong} \tilde{V}, \quad \phi(u_k^j) = \tilde{u}_k^j \ \forall j, k$$

Let

$$\mathcal{E}(\mathbf{v}, \mathbf{w}) = E(\mathbf{v}, \mathbf{w}) / \sim$$

Moduli spaces of representations of $\mathfrak{e}(2)$

Theorem

There is a natural one-to-one correspondence

$$\mathcal{E}(\mathbf{v},\mathbf{w}) \leftrightarrow \mathcal{L}_{Q_\infty}(V,W)$$

if $\mathbf{dim}\, V = \mathbf{v}$, $\mathbf{dim}\, W = \mathbf{w}$

Idea of proof

Given $(V,(u_k^j)) \in E(\mathbf{v},\mathbf{w})$, define a point $(x,s) \in L_{Q_\infty}(V,W)$ by

$$x_{h_i} = p_+|_{V_i}, \quad x_{\bar{h}_i} = p_-|_{V_{i+1}}, \quad k \in \mathbb{Z}$$
$$s(w_k^j) = u_k^j, \quad k \in \mathbb{Z}, \quad 1 \le j \le \mathbf{w}_k$$

where $\{w_k^j\}_{1\le j\le \mathbf{w}_k}$ is a basis of W_k. Then

$$\text{generating set} \leftrightarrow \text{stability} \qquad \sim\leftrightarrow G_V - \text{orbits}$$

Remarks

- relationship between rep theory of Euclidean group and rep theory of $\mathfrak{sl}_\infty$ (or groups $SL(n)$)
- moduli space of reps of Euclidean group along with a set of generators closely related to rep theory of $\mathfrak{sl}_\infty$ and $SL(n)$
- although Euclidean group has wild rep type, we have method of approaching classification:
  - fix cardinality and weights of a generating set
  - resulting moduli space enumerated by countable number of varieties – one variety for reps of each graded dimension

Further directions

Positive characteristic

- consider Euclidean group over field of characteristic p
- weights now lie in $\mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{Z}$
- category of reps equivalent to category of reps of preprojective algebra of quiver of affine type $\hat{A}_{p-1}$
- quiver varieties related to moduli spaces of solutions to anti-self-dual Yang-Mills equations and Hilbert schemes of points in $\mathbb{C}^2$

Further directions

Crystals and Jordan-Hölder decompositions

- can define crystal structure on set of irred comps of Lusztig and Nakajima quiver varieties
- each irred comp can be identified with a sequence of crystal operators
- sequence corresponds to Jordan-Hölder decomposition of $\mathfrak{e}(2)$-modules