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Journal of Algebra 269 (2003) 544–555
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Quiver varieties and fusion products for $\mathfrak{sl}_2$

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Received 31 May 2002

Communicated by Michel van den Bergh

Introduction

In a remarkable series of work starting in [N1], Nakajima gives a geometric realization of integrable highest weight representations V_λ of a Kac–Moody algebra $\mathfrak{g}$ in the homology of a certain Lagrangian subvariety $\mathcal{L}(\lambda)$ of a symplectic variety $\mathcal{M}(\lambda)$ constructed from the Dynkin diagram of $\mathfrak{g}$ (the *quiver variety*). In particular, in [N3], he realizes the tensor product $V_\lambda \otimes V_\mu$ as the homology of a "tensor product variety" $\mathcal{L}(\lambda,\mu) \subset \mathcal{M}(\lambda+\mu)$ (the same construction also appears independently in [M]). When $\mathfrak{g}$ is simple, one might ask if a similar construction can produce the *fusion* tensor products $V_\lambda \otimes_l V_\mu$, certain truncations of $V_\lambda \otimes V_\mu$.

In this short note, we answer this question affirmatively when $\mathfrak{g} = \mathfrak{sl}_2$. In this case, $V_\lambda \otimes_l V_\mu$ is realized as the homology of the most natural subvarieties $\mathcal{L}_l(\lambda,\mu) \subset \mathcal{L}(\lambda,\mu)$ (see Section 3). We also consider the case of a tensor product of arbitrarily many $\mathfrak{sl}_2$-modules $V_{\lambda_1}, \ldots, V_{\lambda_r}$. Finally, we give a combinatorial description of the irreducible components of $\mathcal{L}_l(\lambda,\mu)$ (and $\mathcal{L}_l(\lambda_1,\ldots,\lambda_r)$) using the notions of graphical calculus and crossingless matches for $\mathfrak{sl}_2$ (see [FK] and [S]). We do not expect these constructions to generalize to Lie algebras of higher rank.

1. Fusion products for $U(\mathfrak{sl}_2)$

1.1. Let $\mathcal{R}$ denote the category of finite-dimensional $\mathfrak{sl}_2$-modules, and for $i \geqslant 0$ let V_i denote the simple module of highest weight i. Let $\mathbb{C}[\mathcal{R}]$ be the Grothendieck ring of $\mathcal{R}$

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doi:10.1016/S0021-8693(03)00414-9

and let $[V]$ denote the class of a module V. We have

$$V_i \otimes V_j \simeq \bigoplus_{k=j-i}^{i+j} V_k, \qquad [V_i]\cdot[V_j] = \sum_{k=j-i}^{i+j} [V_k], \quad \text{for } i \leqslant j,$$

where in the sums k increases by twos.

1.2. Now let us fix some positive integer $l \in \mathbb{N}$. Consider the quotient

$$\mathbb{C}_l[\mathcal{R}] = \mathbb{C}[\mathcal{R}]/[V_{l+1}]\mathbb{C}[\mathcal{R}].$$

Denoting by $[V]_l$ the image of $[V]$ in $\mathbb{C}_l[\mathcal{R}]$, we have $\mathbb{C}_l[\mathcal{R}] = \mathbb{C}[V_0]_l \oplus \cdots \oplus \mathbb{C}[V_l]_l$, and

$$[V_i \otimes V_j]_l = \sum_{k=j-i}^{\min(i+j,2l-i-j)} [V_k]_l, \quad \text{for } i \leqslant j \leqslant l.$$

We also set

$$V_i \otimes_l V_j = \bigoplus_{k=j-i}^{\min(i+j,2l-i-j)} V_k, \quad \text{for } i \leqslant j \leqslant l.$$

Again, in the above sums, k increases by twos. The ring $\mathbb{C}_l[\mathcal{R}]$ appears in conformal field theory (as the Grothendieck ring of the modular category of integrable $\widehat{\mathfrak{sl}}_2$-modules of level l) and in quantum group theory (as the Grothendieck ring of a suitable quotient of the category of tilting modules over $U_\epsilon(\mathfrak{sl}_2)$ when ϵ is an lth root of unity).

2. Lagrangian construction of $U(\mathfrak{sl}_2)$

We briefly recall Ginzburg's construction of irreducible representations of $\mathfrak{sl}_2$ in the homology of certain varieties associated to partial flag varieties (cf. [G]). We use the (in this case equivalent) language of quiver varieties (cf. [N2]).

2.1. Let $v, w \in \mathbb{N}$ and let V and W be $\mathbb{C}$-vector spaces of dimensions v and w, respectively. Consider the space

$$M(v,w) = \big\{(i,j) \mid ij = 0;\ \ker j = \{0\}\big\} \subset \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W).$$

We let $\mathrm{GL}(V)$ act on $M(v,w)$ via $g \cdot (i,j) = (gi, jg^{-1})$. This action is free and we set $\mathcal{M}(v,w) = M(v,w)/\mathrm{GL}(V)$. The assignment $(i,j) \mapsto (ji, \operatorname{Im} j)$ defines an isomorphism between $\mathcal{M}(v,w)$ and the variety

$$\mathcal{F}_{v,w} = \big\{(t, V_0) \mid V_0 \subset W,\ \dim V_0 = v,\ \operatorname{Im} t \subset V_0 \subset \ker t\big\} \subset \mathcal{N}_W \times \mathrm{Gr}(v,w),$$

where $\mathcal{N}_W$ is the nullcone of $\mathfrak{gl}(W)$ and $\mathrm{Gr}(v,w)$ is the Grassmannian of v-dimensional subspaces in W. We will denote by $\pi:\mathcal{M}(v,w)\to\mathcal{N}_W$, the projection $(i,j)\mapsto ji$. For any $t\in\mathcal{N}_W$ such that $t^2=0$ we set $\mathcal{M}(v,w)_t=\pi^{-1}(t)$ and $\mathcal{M}(w)_t=\bigsqcup_v\mathcal{M}(v,w)_t$. In particular, we set $\mathcal{L}(v,w)=\pi^{-1}(0)$. Observe that $\mathcal{L}(v,w)$ is just $\mathrm{Gr}(v,w)$ and that $\mathcal{M}(v,w)$ is isomorphic to the cotangent bundle of $\mathcal{L}(v,w)$. We have $\dim\mathcal{M}(v,w)=2\dim\mathcal{L}(v,w)=2v(w-v)$. For $v_1,v_2,w\in\mathbb{N}$ we also consider the variety of triples

$$Z(v_1,v_2,w)=\bigl\{\bigl((i_1,j_1),(i_2,j_2)\bigr)\bigm| j_1i_1=j_2i_2\bigr\}\subset\mathcal{M}(v_1,w)\times\mathcal{M}(v_2,w).$$

Then $\dim Z(v_1,v_2,w)=v_1(w-v_1)+v_2(w-v_2)$.

The form $\omega((i,j),(i',j'))=\mathrm{Tr}_V(ij'-i'j)$ defines a symplectic structure on $\mathcal{M}(v,w)$, for which the variety $\mathcal{L}(v,w)$ is Lagrangian. Equip $\mathcal{M}(v_1,w)\times\mathcal{M}(v_2,w)$ with the symplectic form $\omega\times(-\omega)$. Then $Z(v_1,v_2,w)$ is also Lagrangian. Let denote $Z(w)=\bigsqcup_{v_1,v_2}Z(v_1,v_2,w)$.

2.2. For any complex algebraic variety X we let $H_*(X)$ be the Borel–Moore homology with coefficients in $\mathbb{C}$, and set $H_{\mathrm{top}}(X)=H_{2d}(X)$ where $d=\dim X$.

Let $p_{ij}:\mathcal{M}(v_1,w)\times\mathcal{M}(v_2,w)\times\mathcal{M}(v_3,w)\to\mathcal{M}(v_i,w)\times\mathcal{M}(v_j,w)$ be the obvious projections. The map

$$p_{13}:p_{12}^{-1}\bigl(Z(v_1,v_2,w)\bigr)\cap p_{23}^{-1}\bigl(Z(v_2,v_3,w)\bigr)\to Z(v_1,v_3,w)$$

is proper and we can define the convolution product

$$\begin{aligned}H_i\bigl(Z(v_1,v_2,w)\bigr)\otimes H_j\bigl(Z(v_2,v_3,w)\bigr)&\to H_{i+j-d_2}\bigl(Z(v_1,v_3,w)\bigr),\\ c\otimes c'&\mapsto p_{13*}\bigl(p_{12}^*(c)\cap p_{23}^*(c')\bigr),\end{aligned}$$

where $d_2=4v_2(w-v_2)$. In particular, this gives rise to an algebra structure on $H_{\mathrm{top}}(Z(w))=\bigoplus_{v_1,v_2}H_{\mathrm{top}}(Z(v_1,v_2,w))$.

Now let $t\in\mathcal{N}_W$ such that $t^2=0$. The projection

$$p_1:Z(v_1,v_2,w)\cap p_2^{-1}\bigl(\mathcal{M}(v_2,w)_t\bigr)\to\mathcal{M}(v_1,w)_t$$

(where p_1 and p_2 are the obvious projections) is proper and the convolution action

$$\begin{aligned}H_{\mathrm{top}}\bigl(Z(v_1,v_2,w)\bigr)\otimes H_{\mathrm{top}}\bigl(\mathcal{M}(v_2,w)_t\bigr)&\to H_{\mathrm{top}}\bigl(\mathcal{M}(v_1,w)_t\bigr),\\ c\otimes c'&\mapsto p_{1*}\bigl(c\cap p_2^*(c')\bigr)\end{aligned}$$

makes $H_{\mathrm{top}}(\mathcal{M}(w)_t)=\bigoplus_v H_{\mathrm{top}}(\mathcal{M}(v,w)_t)$ into a $H_{\mathrm{top}}(Z(w))$-module.

Theorem [G]. *There is a natural surjective homomorphism $\Phi:U(\mathfrak{sl}_2)\to H_{\mathrm{top}}(Z(w))$. Under Φ, the module $H_{\mathrm{top}}(\mathcal{M}(w)_t)$ is isomorphic to V_{w-2u} where $u=\mathrm{rank}\,t$.*

2.3. We now give the realization of tensor products of $U(\mathfrak{sl}_2)$-modules. Let $w = w_1 + \cdots + w_r$ and fix $W = W_1 \oplus \cdots \oplus W_r$ with $\dim W_i = w_i$. Let $W_0 = 0$. The group $\mathrm{GL}(W)$ acts on $\mathcal{M}(v,w)$ by $g \cdot (i,j) = (ig^{-1}, gj)$. Consider the embedding

$$\sigma : (\mathbb{C}^*)^{r-1} \to \prod_{i=1}^{r} \mathrm{GL}(W_i) \subset \mathrm{GL}(W),$$
$$(t_2, t_3, \ldots, t_r) \mapsto \left(\mathrm{Id}, t_2^{-1}, t_2^{-1}t_3^{-1}, \ldots, t_2^{-1} \cdots t_r^{-1}\right).$$

Then, for each v, we have (see, e.g, [N3, Lemma 3.2])

$$\mathcal{M}(v,w)^\sigma = \bigsqcup_{v_1+\cdots+v_r=v} \mathcal{M}(v_1,w_1) \times \cdots \times \mathcal{M}(v_r,w_r).$$

Consider the subvarieties

$$\mathcal{M}(v,w_1,\ldots,w_r) = \left\{x \in \mathcal{M}(v,w) \;\middle|\; \lim_{t_i \to 0} \sigma(t_2,\ldots,t_r) \cdot x \text{ exists}\right\},$$
$$\mathcal{N}_W(w_1,\ldots,w_r) = \left\{t \in \mathcal{N}_W \;\middle|\; \lim_{t_i \to 0} \sigma(t_2,\ldots,t_r) \cdot t \text{ exists}\right\}.$$

For $x \in \mathcal{M}(v,w_1,\ldots,w_r)$, let us set $\tau(x) = \lim_{t_i \to 0} \sigma(t_2,\ldots,t_r) \cdot x$. We define $\tau(t)$ similarly for $t \in \mathcal{N}_W(w_1,\ldots,w_r)$. Now consider

$$\mathcal{L}(v,w_1,\ldots,w_r) = \left\{x \in \mathcal{M}(v,w_1,\ldots,w_r) \;\middle|\; \tau(x) \in \prod_i \mathcal{L}(v_i,w_i) \text{ for some } (v_i)\right\}.$$

Set $\mathcal{L}(w_1,\ldots,w_r) = \bigsqcup_v \mathcal{L}(v,w_1,\ldots,w_r)$. Note that $\mathcal{L}(w_1,\ldots,w_r) = \pi^{-1}(\tau^{-1}(0))$ so that we have an action of $H_{\mathrm{top}}(Z(w))$ on $H_{\mathrm{top}}(\mathcal{L}(w_1,\ldots,w_r))$. Moreover, it is easy to check that $\mathcal{L}(w_1,\ldots,w_r)$ is Lagrangian. Note that $\mathcal{L}(w_1,\ldots,w_r)$ is isomorphic to the variety

$$\left\{(t,V_0) \;\middle|\; V_0 \subset W,\ \mathrm{Im}\, t \subset V_0 \subset \ker t,\ t(W_j) \subset W_0 \oplus \cdots \oplus W_{j-1},\ 1 \leqslant j \leqslant r\right\}.$$

Theorem [GRV,N3,M]. *$H_{\mathrm{top}}(\mathcal{L}(w_1,\ldots,w_r))$ is isomorphic to $V_{w_1} \otimes \cdots \otimes V_{w_r}$ as a $U(\mathfrak{sl}_2)$-module.*

3. Lagrangian construction of the fusion product

Let us fix some positive integer l. We will now describe an open subvariety of $\mathcal{L}(w_1,\ldots,w_r)$ whose homology realizes the fusion product $V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}$.

3.1. We keep the notation of Section 2.3. For all $k \in \mathbb{N}$ and $t \in \mathcal{N}_{W_1\oplus\cdots\oplus W_k}(w_1,\dots,w_k)$ we set $\tau_k(t) = \lim_{t_k\to 0}\sigma(1,\dots,1,t_k)(t)$. Let us consider the open subvariety $\mathcal{N}^l(w_1, w_2) = \{t \in \mathcal{N}_{W_1\oplus W_2} \mid \dim\ker t \leqslant l\}$ of $\mathcal{N}_{W_1\oplus W_2}$ and define inductively

$$\mathcal{N}^l(w_1,\dots,w_k) = \bigl\{t \in \mathcal{N}_{W_1\oplus\cdots\oplus W_k} \bigm| \dim\ker t \leqslant l + \operatorname{rank}\tau_k(t),\ t|_{W_1\oplus\cdots\oplus W_{k-1}} \in \mathcal{N}^l(w_1,\dots,w_{k-1})\bigr\} \tag{3.1}$$

for $k \geqslant 3$. Finally, set $\mathcal{L}_l(w_1,\dots,w_r) = \mathcal{L}(w_1,\dots,w_r)\cap\pi^{-1}(\mathcal{N}^l(w_1,\dots,w_r))$. By definition $\mathcal{L}_l(w_1,\dots,w_r)$ is an open subvariety of $\mathcal{L}(w_1,\dots,w_r)$ and therefore $H_{\mathrm{top}}(\mathcal{L}_l(w_1,\dots,w_r))$ is a $H_{\mathrm{top}}(Z(w))$-module.

Theorem. *$H_{\mathrm{top}}(\mathcal{L}_l(w_1,\dots,w_r))$ is isomorphic to $V_{w_1}\otimes_l\cdots\otimes_l V_{w_r}$ as a $U(\mathfrak{sl}_2)$-module.*

Proof. We proceed by induction. Suppose $r = 2$. It is enough to describe the irreducible components of $\mathcal{L}_l(w_1, w_2)$ corresponding to highest weight vectors in the $U(\mathfrak{sl}_2)$-module $H_{\mathrm{top}}(\mathcal{L}_l(w_1, w_2))$. The irreducible components of $\mathcal{L}(w_1, w_2)$ corresponding to highest-weight vectors are

$$I_v = \bigl\{(i,j) \bigm| j(V)\subset W_1,\ i(W_2) = V,\ i(W_1) = 0\bigr\}, \quad \text{for } 0 \leqslant v \leqslant w_1, w_2,$$

and the associated highest weight is $w_1 + w_2 - 2v$. Note that the condition $\dim\ker ji \leqslant l$ is equivalent to the condition $w_1 + w_2 - 2v \leqslant 2l - w_1 - w_2$. Now suppose that the theorem is proved for tensor products of $r - 1$ modules, and let us set $W' = W_1 \oplus \cdots \oplus W_{r-1}$. For each $u \in \mathbb{N}$ let us set $\mathcal{N}_{W'}(u) = \{t \in \mathcal{N}_{W'} \mid \operatorname{rank} t = u\}$. Recall that $\mathcal{L}_l(w_1,\dots,w_{r-1})$ is Lagrangian and that π is semi-small with all strata being relevant (cf. [N2, Section 10]). Thus $\pi(\mathcal{L}_l(w_1,\dots,w_{r-1}))\cap\mathcal{N}_{W'}(u)$ is a subvariety of $\mathcal{N}_{W'}(u)$ of dimension $\frac{1}{2}\dim\mathcal{N}_{W'}(u)$. Let $\mathcal{C}^u_1,\dots,\mathcal{C}^u_{s(u)}$ be its irreducible components. By the induction hypothesis,

$$s(u) = \dim\operatorname{Hom}_{\mathfrak{sl}_2}(V_{w'-2u}, V_{w_1}\otimes_l\cdots\otimes_l V_{w_{r-1}}). \tag{3.2}$$

The irreducible components of $\mathcal{L}_l(v, w_1,\dots,w_r)$ corresponding to highest weight vectors of $H_{\mathrm{top}}(\mathcal{L}_l(w_1,\dots,w_r))$ are of the form $\overline{I_\chi}$ with

$$I_\chi = \bigl\{(i,j) \bigm| i(W) = V,\ j(V)\subset W',\ (i_{W'}, j) \in \chi\bigr\},$$

where χ is an irreducible component of $\mathcal{L}_l(v, w_1,\dots,w_{r-1})$, and the associated highest weight is $w - 2v$ (note that I_χ may be empty). Let us fix $u \in \mathbb{N}$ and $\mathcal{C}^u_k$ for some $k \leqslant s(u)$. Let $\chi \subset \overline{\pi^{-1}(\mathcal{C}^u_k)}\cap\mathcal{L}_l(v, w_1,\dots,w_{r-1})$ be an irreducible component. Then $I_\chi \subset \overline{\mathcal{L}_l(w_1,\dots,w_r)}$ if for all (i,j) in (an open dense subset of) I_χ we have $\dim\operatorname{Im} ji \leqslant l + u$. This is equivalent to the condition that the corresponding highest weight $w - 2v$ satisfies

$$w - 2v \leqslant 2l - w_r - (w' - 2u). \tag{3.3}$$

Eqs. (3.2) and (3.3) together imply that

$$H_{\text{top}}\big(\mathcal{L}_l(w_1,\dots,w_r)\big) \simeq (V_{w_1} \otimes_l \cdots \otimes_l V_{w_{r-1}}) \otimes_l V_{w_r}$$

as a $U(\mathfrak{sl}_2)$-module, as desired. □

Remark. (i) The above construction is not canonical in the sense that it was made using a choice of a bracketing of the tensor product, namely

$$\big(\cdots\big((V_{w_1} \otimes_l V_{w_2}) \otimes_l V_{w_3}\big)\cdots \otimes_l V_{w_r}\big).$$

Different bracketings give rise to different (possibly non-isomorphic) open subvarieties of $\mathcal{L}_l(w_1,\dots,w_r)$ realizing the same fusion tensor product.

(ii) One might be tempted to define in an analogous fashion a truncated tensor product for finite-dimensional representations of $U_q(\widehat{\mathfrak{sl}}_2)$ by considering equivariant K-theory of $\mathcal{L}_l(w_1,w_2)$ rather than Borel–Moore homology. However, it is easy to check that (because of Remark (i)) the resulting product is not associative.

4. A graphical calculus for the fusion product

4.1. We first recall some results on the graphical calculus of tensor products and intertwiners. For a more complete treatment, see [FK,S]. In the graphical calculus, V_d is depicted by a box marked d with d vertices. To depict the set $CM^{\mu}_{w_1,\dots,w_r}$ of crossingless matches, we place the boxes representing the V_{w_i} on a horizontal line and the box representing V_μ on another horizontal line lying above the first one. $CM^{\mu}_{w_1,\dots,w_r}$ is then the set of non-intersecting curves (up to isotopy) connecting the vertices of the boxes such that the following conditions are satisfied:

1. Each curve connects exactly two vertices.
2. Each vertex is the end point of exactly one curve.
3. No curve joins a box to itself.
4. The curves lie inside the box bounded by the two horizontal lines and the vertical lines through the extreme right and left points.

We call the curves joining two lower boxes *lower curves* and those joining a lower and an upper box *middle curves*. We define the set of oriented crossingless matches $OCM^{\mu}_{w_1,\dots,w_r}$ to be the set of elements of $CM^{\mu}_{w_1,\dots,w_r}$ along with an orientation of the curves such that all lower curves are oriented to the left and all middle curves are oriented so that those oriented down are to the right of those oriented up.

As shown in [FK], the set of crossingless matches $CM^{\mu}_{w_1,\dots,w_r}$ is in one-to-one correspondence with a basis of the set of intertwiners

$$H^{\mu}_{w_1,\dots,w_r} \stackrel{\text{def}}{=} \operatorname{Hom}(V_{w_1} \otimes \cdots \otimes V_{w_r}, V_\mu).$$

The matrix coefficients of the intertwiner associated to a particular crossingless match are given by Theorem 2.1 of [FK].

We will also need to define the set of *lower crossingless matches* $LCM^{\mu}_{w_1,\dots,w_r}$ and *oriented lower crossingless matches* $OLCM^{\mu}_{w_1,\dots,w_r}$. Elements of $LCM^{\mu}_{w_1,\dots,w_r}$ and $OLCM^{\mu}_{w_1,\dots,w_r}$ are obtained from elements of $CM^{\mu}_{w_1,\dots,w_r}$ and $OCM^{\mu}_{w_1,\dots,w_r}$ (respectively) by removing the upper box (thus converting lower end points of middle curves to unmatched vertices). For the case of $OLCM^{\mu}_{w_1,\dots,w_r}$, unmatched vertices will still have an orientation (indicated by an arrow attached to the vertex). As for middle curves in the case of $OCM^{\mu}_{w_1,\dots,w_r}$, the unmatched vertices in an element of $OLCM^{\mu}_{w_1,\dots,w_r}$ must be arranged so that those oriented down are to the right of those oriented up.

Note that the set of lower crossingless matches $LCM = LCM_{w_1,\dots,w_r}$ is in one-to-one correspondence with the set $\bigcup_{\mu} CM^{\mu}_{w_1,\dots,w_r}$. From now on, we will identify these two sets.

4.2. Let s be a bracketing of the tensor product $V_{w_1} \otimes \cdots \otimes V_{w_r}$. Pick an ordering of the tensor operations compatible with this bracketing. For each n such that $1 \leqslant n \leqslant r-1$, let S_n be the set of the V_{w_i} separated from the nth tensor product operation only by operations ranked lower than or equal to n. Then let ${}^{l}_{s}CM^{\mu}_{w_1,\dots,w_r}$ be the set of elements of $CM^{\mu}_{w_1,\dots,w_r}$ satisfying the following condition: for each n, the number of curves connecting V_{w_i}'s in S_n to either V_{w_i}'s in S_n on the other side of the nth tensor product symbol or V_w's not in S_n is less than or equal to l. Note that this condition does not depend on the particular ordering so long as it is compatible with the bracketing s.

Let ${}^{l}_{s}LCM = {}^{l}_{s}LCM_{w_1,\dots,w_r}$ be the set of lower crossingless matches satisfying the same condition (where unmatched vertices are always counted as curves with the other end point outside of any S_n) and identify this set with the set $\bigcup_{\mu} {}^{l}_{s}CM^{\mu}_{w_1,\dots,w_r}$. We define ${}^{l}_{s}OCM^{\mu}_{w_1,\dots,w_r}$ and ${}^{l}_{s}OLCM = {}^{l}_{s}OLCM_{w_1,\dots,w_r}$ similarly (and the corresponding identification is made).

Note that in the case $r = 2$ the condition in the definition simplifies to the requirement that the total number of curves (including middle curves) is less than or equal to l. In fact, the given definition simply arises from applying this condition to each tensor product operation (in the given ordering), neglecting curves with both end points in V_{w_i}'s which have already been tensored together.

Proposition. *The set ${}^{l}_{s}CM^{\mu}_{w_1,\dots,w_r}$ is in one-to-one correspondence with a basis of the space of intertwiners ${}^{l}H^{\mu}_{w_1,\dots,w_r} \stackrel{\text{def}}{=} \mathrm{Hom}(V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}, V_{\mu})$.*

Proof. We first consider the case $r = 2$. For any $b \in CM^{\mu}_{w_1,w_2}$, the total number of curves is equal to $(w_1 + w_2 + \mu)/2$ (since each vertex is an end point of exactly one curve). Thus the condition that the total number of curves is less than or equal to l reduces to $w_1 + w_2 + \mu \leqslant 2l$ or $\mu \leqslant 2l - w_1 - w_2$ as desired.

Now assume the result holds for the product of less than r irreducible modules and that for the product of V_{w_1} through V_{w_r}, the rth tensor product operation is the one occurring between V_{w_k} and $V_{w_{k+1}}$ $(k < r)$. Note that

$$\bigoplus_{\nu} {}^{l}H^{\nu}_{w_1,\dots,w_k} \otimes {}^{l}H^{\mu}_{\nu,w_{k+1},\dots,w_r} \cong {}^{l}H^{\mu}_{w_1,\dots,w_r}$$

via the map $f \otimes g \mapsto g(f \otimes \mathrm{id}_{V_{w_{k+1}} \oplus \cdots \oplus V_{w_r}})$. Now, if s_1 is the bracketing of the first k modules and s_2 is the bracketing of the last $r-k$ modules, it is easy to see that

$$\sum_{\nu} {}_{s_1}^{\;l}CM^{\nu}_{w_1,\dots,w_k} \times {}_{s_2}^{\;l}CM^{\mu}_{\nu,w_{k+1},\dots,w_r} \cong {}_{s}^{l}CM^{\mu}_{w_1,\dots,w_r} \quad \text{(as sets)}.$$

The result now follows by induction. □

4.3. From the associativity of the fusion tensor product it follows immediately that the order of the set ${}_{s}^{l}CM^{\mu}_{w_1,\dots,w_r}$ is independent of the bracketing s. However, we will present here a direct proof.

Proposition. *The order of the set ${}_{s}^{l}CM^{\mu}_{w_1,\dots,w_r}$ is independent of the bracketing s.*

Proof. It suffices to prove the statement for three factors. Let s_1 be the bracketing $(V_{w_1} \otimes V_{w_2}) \otimes V_{w_3}$ and s_2 be the bracketing $V_{w_1} \otimes (V_{w_2} \otimes V_{w_3})$. We will set up a one-to-one correspondence between ${}_{s_1}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ and ${}_{s_2}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$. We will first establish a one-to-one correspondence between the subsets consisting of those crossingless matches with no curves connecting V_{w_1} and V_{w_3} and a fixed number n of lower curves. Let a (respectively b) denote the number of curves connecting V_{w_1} (respectively V_{w_3}) to V_{w_2}. Thus $a+b=n$. Now, the number of curves with at least one end point in V_{w_1} or V_{w_2} is w_1+w_2-a and the total number of curves minus the curves connecting V_{w_1} to V_{w_2} is $w_1+w_2+w_3-n-a$. Thus a crossingless match lies in ${}_{s_1}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ if and only if

$$w_1+w_2-a \leqslant l, \qquad w_1+w_2+w_3-n-a \leqslant l.$$

Similarly, a crossingless match lies in ${}_{s_2}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ if and only if

$$w_2+w_3-b \leqslant l, \qquad w_1+w_2+w_3-n-b \leqslant l.$$

Now, the largest possible value of a is $\min(w_1,n)$ and the largest possible value of b is $\min(w_3,n)$. Therefore, by counting the possible values of a, the number of crossingless matches in ${}_{s_1}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ with no curves connecting V_{w_1} and V_{w_3} and with n total curves is equal to

$$r_a = \min(w_1,n) - \max(w_1+w_2-l,\, w_1+w_2+w_3-n-l) + 1$$

if this number is positive and zero otherwise. Similarly, the number of crossingless matches in ${}_{s_2}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ with no curves connecting V_{w_1} and V_{w_3} and with n total curves is equal to

$$r_b = \min(w_3,n) - \max(w_2+w_3-l,\, w_1+w_2+w_3-n-l) + 1$$

if this number is positive and zero otherwise. Considering the four cases $n \leqslant w_1, w_3$; $n \geqslant w_1, w_3$; $w_1 \leqslant n \leqslant w_3$ and $w_3 \leqslant n \leqslant w_1$ we easily see that $r_a = r_b$ in all cases.

It remains to establish a one-to-one correspondence between the elements of ${}_{s_1}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ and ${}_{s_2}^{\;l}CM^{\mu}_{w_1,\dots,w_r}$ with $c \geqslant 1$ curves joining V_{w_1} and V_{w_3}. Fix the number of lower curves with one end point in V_{w_2} to be n. Since V_{w_1} and V_{w_3} are connected, there can be no middles curves with end points in V_{w_2}. Thus $s = w_2$. Define a and b as above. By an argument analogous to that given in the earlier case, the number of crossingless matches in ${}^{\,l}_{s_1}CM^{\mu}_{w_1,\dots,w_r}$ with $c \geqslant 1$ curves connecting V_{w_1} to V_{w_3} and with n lower curves with one end point in V_{w_2} is equal to

$$r_a = \min(w_1 - c, w_2) - \max(w_1 + w_2 - l, w_1 + w_3 - l - c) + 1$$

if this number is positive and zero otherwise. Similarly, the number of crossingless matches in ${}^{\,l}_{s_2}CM^{\mu}_{w_1,\dots,w_r}$ with $c \geqslant 1$ curves connecting V_{w_1} to V_{w_3} and with n lower curves with one end point in V_{w_2} is equal to

$$r_b = \min(w_3 - c, w_2) - \max(w_2 + w_3 - l, w_1 + w_3 - l - c) + 1$$

if this number is positive and zero otherwise. Considering the four cases: $w_2 \leqslant w_1 - c, w_3 - c$; $w_2 \geqslant w_1 - c, w_3 - c$; $w_1 - c \leqslant w_2 \leqslant w_3 - c$ and $w_3 - c \leqslant w_2 \leqslant w_1 - c$, we easily see that $r_a = r_b$ in all cases. This concludes the proof. □

From now on, we will use the bracketing $(\cdots((V_{w_1} \otimes V_{w_2}) \otimes V_{w_3}) \cdots V_{w_r})$ unless explicitly stated otherwise. Thus, if we omit a subscript s, we take s to be this bracketing.

5. The fusion product via constructible functions

5.1. Fix a $w = w_1 + \cdots + w_r$ dimensional $\mathbb{C}$-vector space W and let

$$\begin{aligned}\mathfrak{T}(w_1, \dots, w_r) = \big\{ \big(\mathbf{D} = \{D_i\}_{i=0}^r, V_0, t\big) \bigm| 0 = D_0 \subset D_1 \subset \cdots \subset D_r = W,\ V_0 \subset W, \\ t \in \operatorname{End} W,\ t(D_i) \in D_{i-1},\ \dim(D_i/D_{i-1}) = w_i,\ \operatorname{Im} t \subset V_0 \subset \ker t \big\}.\end{aligned}$$

Consider the projection

$$\mathfrak{T}(w_1, \dots, w_r) \to \big\{ \mathbf{D} = \{D_i\}_{i=0}^r \bigm| 0 = D_0 \subset D_1 \subset \cdots \subset D_r = W,\ \dim(D_i/D_{i-1}) = w_i \big\}$$

given by $(\mathbf{D}, V_0, t) \mapsto \mathbf{D}$. It is easy to see that the fibers of this map are all isomorphic and that in [S] one could replace the tensor product variety $\mathfrak{T}(w_1, \dots, w_r)$ by this fiber, restrict the constructible functions to this fiber and the theory would remain unchanged. Let $\mathfrak{T}_{\mathbf{D}}(w_1, \dots, w_r)$ denote the fiber over a flag $\mathbf{D}$. If we define

$$D_i = W_0 \oplus \cdots \oplus W_i, \quad 0 \leqslant i \leqslant r,$$

then obviously

$$\mathfrak{T}_{\mathbf{D}}(w_1, \dots, w_r) \cong \mathcal{L}(w_1, \dots, w_r)$$

and in the sequel we will identify these two varieties.

5.2. If $b \in CM^{\mu}_{w_1,\dots,w_r}$ is an unoriented crossingless match, let

$$Y_b = \bigl\{(\mathbf{D}, V_0, t) \in \mathfrak{T}(w_1, \dots, w_r) \bigm| \dim(\ker t \cap D_i)/(\ker t \cap D_{i-1}) = b_i\bigr\},$$

where b_i is the number of left end points (of lower curves) and lower end points (of middle curves) contained in the box representing V_{w_i}. It is shown in [S] (Proposition 3.2.1) that $\bigsqcup_b Y_b = \mathfrak{T}(w_1, \dots, w_r)$ and that the closures of the Y_b are precisely the irreducible components of $\mathfrak{T}(w_1, \dots, w_r)$. Let $X_b = Y_b \cap \mathcal{L}(w_1, \dots, w_r)$. Then obviously $\mathcal{L}(w_1, \dots, w_r) = \bigsqcup_{b \in LCM} X_b$.

Proposition. $\mathcal{L}_l(w_1, \dots, w_r) = \bigsqcup_{b \in {}^l LCM} X_b$.

Proof. We see from Eq. (3.1) that $\mathcal{L}_l(w_1, \dots, w_r)$ is the set of all $(t, V_0) \in \mathcal{L}(w_1, \dots, w_r)$ such that

$$\dim \ker t|_{W_1 \oplus \dots \oplus W_i} \leqslant l + \operatorname{rank} t|_{W_1 \oplus \dots \oplus W_{i-1}} \quad \forall 1 \leqslant i \leqslant r.$$

Now, by the definition of the X_b, if $(t, V_0) \in X_b$ for some $b \in LCM$ then $\dim \ker t|_{W_1 \oplus \dots \oplus W_i}$ is equal to $\sum_{j=1}^{i} w_j$ minus the number of lower curves with both end points among the lower i boxes. Also, $\operatorname{rank} t|_{W_1 \oplus \dots \oplus W_{i-1}}$ is equal to the number of lower curves with both end points among the lower $i-1$ boxes. Let c_i denote the number of curves with both end points among the lower i boxes. Then

$$\begin{aligned}
&\dim \ker t|_{W_1 \oplus \dots \oplus W_i} \leqslant l + \operatorname{rank} t|_{W_1 \oplus \dots \oplus W_{i-1}} \\
&\quad \Leftrightarrow \quad \sum_{j=1}^{i} w_j - c_i \leqslant l + c_{i-1} \\
&\quad \Leftrightarrow \quad \sum_{j=1}^{i} w_j - 2c_{i-1} - \#\{\text{curves with right end point in } i\text{th box}\} \leqslant l \\
&\quad \Leftrightarrow \quad \sum_{i=1}^{n} w_i - \#\{\text{end points in first } i-1 \text{ boxes of lower curves with both} \\
&\qquad\qquad\qquad \text{end points in first } i \text{ boxes}\} \leqslant l
\end{aligned}$$

and this is easily seen to be equivalent to the condition that $b \in {}^l LCM$ (with the default bracketing). □

5.3. We will now define a $U(\mathfrak{sl}_2)$-module structure on a certain space of constructible functions on $\mathcal{L}_l(w_1, \dots, w_r)$. For $\mathbf{a} \in OLCM_{w_1,\dots,w_r}$, let $\bar{\mathbf{a}}$ be the associated element of $LCM_{w_1,\dots,w_r}$ obtained by forgetting the orientation. Define

$$Y_{\mathbf{a}} = \bigl\{(\mathbf{D}, V_0, t) \in Y_{\bar{\mathbf{a}}} \bigm| \dim W = \#\{\text{up-oriented vertices of } \mathbf{a}\}\bigr\},$$

where the right end points of lower curves are oriented up (as well as the up-oriented unmatched vertices). Let $X_{\mathbf{a}} = Y_{\mathbf{a}} \cap \mathcal{L}(w_1, \ldots, w_r)$. Then it follows from Eq. (33) of [S] that

$$X_b = \bigcup_{\mathbf{a}:\ \bar{\mathbf{a}}=b} X_{\mathbf{a}}.$$

Now let

$$\mathcal{B}_s^l = \{\mathbf{1}_{Y_{\mathbf{a}}} \mid \mathbf{a} \in {}^l OLCM\},$$

where $\mathbf{1}_A$ is the function that is equal to one on the set A and zero elsewhere. Let

$$\mathcal{T}^l = \mathcal{T}_s^l(w_1, \ldots, w_r) = \operatorname{Span} \mathcal{B}_s^l.$$

We endow $\mathcal{T}^l$ with the structure of a $U(\mathfrak{sl}_2)$-module as in [S].

Theorem. *$\mathcal{T}_s^l(w_1, \ldots, w_r)$ is isomorphic as a $U(\mathfrak{sl}_2)$-module to $V_{w_1} \otimes_l \cdots \otimes_l V_{w_r}$ and $\mathcal{B}_s^l$ is a basis for $\mathcal{T}_s^l(w_1, \ldots, w_r)$ adapted to its decomposition into a direct sum of irreducible representations. That is, for a given $b \in {}^l CM^{\mu}_{w_1,\ldots,w_r}$, the space $\operatorname{Span}\{\mathbf{1}_{Y_{\mathbf{a}}} \mid \bar{\mathbf{a}} = b\}$ is isomorphic to the irreducible representation V_μ via the map*

$$\mathbf{1}_{Y_{\mathbf{a}}} \mapsto {}^{\mu}v_{\mu - 2\#\{\text{unmatched down-oriented vertices of } \mathbf{a}\}}.$$

Proof. The second part of the theorem follows from Theorem 3.3.1 of [S]. Then

$$\begin{aligned}\mathcal{T}^l &= * \bigoplus_{\mu} \bigoplus_{b \in {}^l CM^{\mu}_{w_1,\ldots,w_r}} \operatorname{Span}\{\mathbf{1}_{Y_{\mathbf{a}}} \mid \bar{\mathbf{a}} = b\} \\ &\cong \bigoplus_{\mu} \bigoplus_{b \in {}^l CM^{\mu}_{w_1,\ldots,w_r}} V_\mu \cong \bigoplus_{\mu} {}^l H^{\mu}_{w_1,\ldots,w_r} \otimes V_\mu \cong V_{w_1} \otimes_l \cdots \otimes_l V_{w_r},\end{aligned}$$

where ${}^l H^{\mu}_{w_1,\ldots,w_r}$ is given the trivial module structure. □

Remarks. We have used here the standard bracketing $(\cdots(V_{w_1} \otimes_l V_{w_2}) \otimes_l V_{w_3}) \cdots \otimes_l V_{w_r})$. However, one could easily modify the definitions to use any other bracketing. The proofs would need only slight changes. Of course, as noted above, while we would still recover the structure of the fusion product, the varieties involved would be non-isomorphic in general.

Acknowledgments

We thank I. Frenkel for raising the question of the relation between quiver varieties and fusion products. The research of the first author was supported in part by the Natural Science and Engineering Research Council of Canada.

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