Braided and coboundary monoidal categories

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Dedicated to Ivan Shestakov on his sixtieth birthday

ABSTRACT. We discuss and compare the notions of braided and coboundary monoidal categories. Coboundary monoidal categories are analogues of braided monoidal categories in which the role of the braid group is replaced by the cactus group. We focus on the categories of representations of quantum groups and crystals and explain how while the former is a braided monoidal category, this structure does not pass to the crystal limit. However, the categories of representations of quantum groups of finite type also possess the structure of a coboundary category which does behave well in the crystal limit. We explain this construction and also a recent interpretation of the coboundary structure using quiver varieties. This geometric viewpoint allows one to show that the category of crystals is in fact a coboundary monoidal category for arbitrary symmetrizable Kac-Moody type.

Introduction

In this expository paper we discuss and contrast two types of categories – braided monoidal categories and coboundary monoidal categories – paying special attention to how the categories of representations of quantum groups and crystals fit into this framework. Monoidal categories are essentially categories with a tensor product, such as the categories of vector spaces, abelian groups, sets and topological spaces. Braided monoidal categories are well-studied in the literature. They are monoidal categories with an action of the braid group on multiple tensor products. The example that interests us the most is the category of representations of a quantum group $U_q(\mathfrak{g})$. Coboundary monoidal categories are perhaps less well known than their braided cousins. The concept is similar, the difference being that the role of the braid group is now played by the so-called *cactus group*. A key component in the definition of a coboundary monoidal category is the *cactus commutor*, which assumes the role of the braiding.

The theory of crystals can be thought of as the $q \to \infty$ (or $q \to 0$) limit of the theory of quantum groups. In this limit, representations are replaced by

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combinatorial objects called crystal graphs. These are edge-colored directed graphs encoding important information about the representations from which they come. Developing concrete realizations of crystals is an active area of research and there exist many different models.

It is interesting to ask if the structure of a braided monoidal category passes to the crystal limit. That is, does one have an induced structure of a braided monoidal category on the category of crystals. The answer is no. In fact, one can prove that it is impossible to give the category of crystals the structure of a braided monoidal category (see Proposition 5.6). However, the situation is more hopeful if one instead considers coboundary monoidal categories. For quantum groups of finite type, there is a way – a unitarization procedure introduced by Drinfel'd [**5**] – to use the braiding on the category of representations to define a cactus commutor on this category. This structure passes to the crystal limit and one can define a coboundary structure on the category of crystals in finite type (see [**7, 10**]). Kamnitzer and Tingley [**9**] gave an alternative definition of the crystal commutor which makes sense for quantum groups of arbitrary symmetrizable Kac-Moody type. However, while this definition agrees with the previous one in finite type, it is not obvious that it satisfies the desired properties, giving the category of crystals the structure of a coboundary category, in other types.

In [**24**], the author gave a geometric realization of the cactus commutor using quiver varieties. In this setting, the commutor turns out to have a very simple interpretation – it corresponds to simply taking adjoints of quiver representations. Equipped with this geometric description, one is able to show that the crystal commutor satisfies the requisite properties and thus the category of crystals, in arbitrary symmetrizable Kac-Moody type, is a coboundary category.

In the current paper, when discussing the topics of quantum groups and crystals, we will often restrict our attention to the quantum group $U_q(\mathfrak{sl}_2)$ and its crystals. This allows us to perform explicit computations illustrating the key concepts involved. The reader interested in the more general case can find the definitions in the references given throughout the paper.

The organization of this paper is as follows. In Section 1, we introduce the braid and cactus groups that play an important role in the categories in which we are interested. In Sections 2 through 4, we define monoidal, braided monoidal, and coboundary monoidal categories. We review the theory of quantum groups and crystals in Section 5. In Section 6 we recall the various definitions of cactus commutors in the categories of representations of quantum groups and crystals. Finally, in Section 7, we give the geometric interpretation of the commutor.

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1. The braid and cactus groups

1.1. The braid group.

DEFINITION 1.1 (Braid group). *For n a positive integer, the n-strand Braid group $\mathcal{B}_n$ is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations*

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for $|i - j| \geq 2$, and*
(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *for $1 \leq i \leq n - 2$.*

These relations are known as the braid relations *and the second is often called the* Yang-Baxter equation.

Recall that the symmetric group S_n is the group on generators $s_1, \ldots, s_{n-1}$ satisfying the same relations as for the σ_i above in addition to the relations $s_i^2 = 1$ for all $1 \leq i \leq n-1$. We thus have a surjective group homomorphism $\mathcal{B}_n \twoheadrightarrow S_n$. The kernel of this map is called the *pure braid group*.

The braid group has several geometric interpretations. The one from which its name is derived is the realization in terms of braids. An *n-strand braid* is an isotopy class of a union of n non-intersecting smooth curve segments (strands) in $\mathbb{R}^3$ with end points $\{1, 2, \ldots, n\} \times \{0\} \times \{0, 1\}$, such that the third coordinate is strictly increasing from 0 to 1 in each strand. The set of all braids with multiplication giving by placing one braid on top of another (and rescaling so that the third coordinate ranges from 0 to 1) is isomorphic to $\mathcal{B}_n$ as defined algebraically above.

The braid group is also isomorphic to the mapping class group of the n-punctured disk – the group of self-homeomorphisms of the punctured disk with n-punctures modulo the subgroup consisting of those homeomorphisms isotopic to the identity map. One can picture the isomorphism by thinking of each puncture being connected to the boundary of the disk by a string. Each homeomorphism of the n-punctured disk can then be seen to yield a braiding of these strings. The pure braid group corresponds to the classes of homeomorphisms that map each puncture to itself.

A similar geometric realization of the braid group is as the fundamental group of the configuration space of n points in the unit disk D. A loop from one configuration to itself in this space defines an n-strand braid where each strand is the trajectory in $D \times [0, 1]$ traced out by one of the n points. If the points are labeled, then we require each point to end where it started and the corresponding fundamental group is isomorphic to the pure braid group.

1.2. The cactus group. Fix a positive integer n. For $1 \leq p < q \leq n$, let

$$\hat{s}_{p,q} = \begin{pmatrix} 1 & \cdots & p-1 & p & p+1 & \cdots & q & q+1 & \cdots & n \\ 1 & \cdots & p-1 & q & q-1 & \cdots & p & q+1 & \cdots & n \end{pmatrix} \in S_n.$$

Since $s_{i,i+1} = s_i$, these elements generate S_n. If $1 \leq p < q \leq n$ and $1 \leq k < l \leq n$, we say that $p < q$ and $k < l$ are *disjoint* if $q < k$ or $l < p$. We say that $p < q$ *contains* $k < l$ if $p \leq k < l \leq q$.

DEFINITION 1.2 (Cactus group). *For n a positive integer, the n-fruit cactus group J_n is the group with generators $s_{p,q}$ for $1 \leq p < q \leq n$ and relations*

(1) $s_{p,q}^2 = 1$,
(2) $s_{p,q}s_{k,l} = s_{k,l}s_{p,q}$ *if $p < q$ and $k < l$ are disjoint, and*
(3) $s_{p,q}s_{k,l} = s_{r,t}s_{p,q}$ *if $p < q$ contains $k < l$, where $r = \hat{s}_{p,q}(l)$ and $t = \hat{s}_{p,q}(k)$.*

It is easily checked that the elements $\hat{s}_{p,q}$ of the symmetric group satisfy the relations defining the cactus group and thus the map $s_{p,q} \mapsto \hat{s}_{p,q}$ extends to a surjective group homomorphism $J_n \twoheadrightarrow S_n$. The kernel of this map is called the *pure cactus group*.

The cactus group also has a geometric interpretation. In particular, the kernel of the surjection $J_n \twoheadrightarrow S_n$ is isomorphic to the fundamental group of the Deligne-Mumford compactification $\overline{M}_0^{n+1}(\mathbb{R})$ of the moduli space of real genus zero curves with $n + 1$ marked points. The generator $s_{p,q}$ of the cactus group corresponds
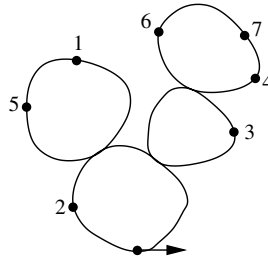
FIGURE 1. A 7-fruited cactus

to a path in $\overline{M}_0^{n+1}(\mathbb{R})$ in which the marked points $p, \ldots, q$ balloon off into a new component, this components flips, and then the component collapses, the points returning in reversed order. Elements of $\overline{M}_0^{n+1}(\mathbb{R})$ look similar to cacti of the genus *Opuntia* (the marked points being flowers) which justifies the name *cactus group* (see Figure 1). Note the similarity with the last geometric realization of the braid group mentioned in Section 1.1. Both the pure braid group and the pure cactus group are fundamental groups of certain spaces with marked points. We refer the reader to [**3, 4, 7**] for further details on this aspect of the cactus group.

1.3. The relationship between the braid and cactus groups. The relationship between the braid group and the cactus group is not completely understood. While the symmetric group is a quotient of both, neither is a quotient of the other. However, there is a homomorphism from the cactus group into the prounipotent completion of the braid group (see the proof of Theorem 3.14 in [**6**]). This map is closely related to the unitarization procedure of Drinfel'd to be discussed in Section 6.1.

In the next few sections, we will define categories that are closely related to the braid and cactus groups. We will see that there are some connections between the two. In addition to the unitarization procedure, we will see that braidings satisfy the so-called cactus relation (see Proposition 4.3).

2. Monoidal categories

2.1. Definitions. Recall that for two functors $F, G : \mathcal{C} \to \mathcal{D}$ a *natural transformation* $\varphi : F \to G$ is a collection of morphisms $\varphi_U : F(U) \to G(U)$, $U \in \mathrm{Ob}\,\mathcal{C}$, such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$, we have $\varphi_V \circ F(f) = G(f) \circ \varphi_U : F(U) \to G(V)$. If the maps φ_U are all isomorphisms, we call φ a *natural isomorphism*. We will sometimes refer to the φ_U themselves as *natural isomorphisms* when the functors involved are clear.

DEFINITION 2.1 (Monoidal Category). *A monoidal category is a category $\mathcal{C}$ equipped with the following:*

(1) *a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the* tensor product,
(2) *natural isomorphisms (the* associator*)*

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

for all $U, V, W \in \mathrm{Ob}\,\mathcal{C}$ satisfying the pentagon axiom*: for all $U, V, W, X \in \mathrm{Ob}\,\mathcal{C}$, the diagram*

$$((U \otimes V) \otimes W) \otimes X$$

$$\alpha_{U,V,W} \otimes \mathrm{id}_X \qquad \qquad \alpha_{U \otimes V, W, X}$$

$$(U \otimes (V \otimes W)) \otimes X \qquad \qquad (U \otimes V) \otimes (W \otimes X)$$

$$\alpha_{U, V \otimes W, X} \qquad \qquad \alpha_{U, V, W \otimes X}$$

$$U \otimes ((V \otimes W) \otimes X) \xrightarrow{\;\;\mathrm{id}_U \otimes \alpha_{V,W,X}\;\;} U \otimes (V \otimes (W \otimes X))$$

commutes, and

(3) *a* unit object $\mathbf{1} \in \mathrm{Ob}\,\mathcal{C}$ *and natural isomorphisms*

$$\lambda_V : \mathbf{1} \otimes V \xrightarrow{\;\cong\;} V, \quad \rho_V : V \otimes \mathbf{1} \xrightarrow{\;\cong\;} V$$

for every $V \in \mathrm{Ob}\,\mathcal{C}$, satisfying the triangle axiom*: for all $U, V \in \mathrm{Ob}\,\mathcal{C}$, the diagram*

$$(U \otimes \mathbf{1}) \otimes V \xrightarrow{\;\;\alpha_{U,\mathbf{1},V}\;\;} U \otimes (\mathbf{1} \otimes V)$$

$$\rho_U \otimes \mathrm{id}_V \qquad \qquad \mathrm{id}_U \otimes \lambda_V$$

$$U \otimes V$$

commutes.

A monoidal category is said to be strict *if we can take $\alpha_{U,V,W}$, λ_V and ρ_V to be identity morphisms for all $U, V, W \in \mathrm{Ob}\,\mathcal{C}$. That is, a* strict monoidal category *is one in which*

$$V \otimes \mathbf{1} = V, \quad \mathbf{1} \otimes V = V, \quad (U \otimes V) \otimes W = U \otimes (V \otimes W)$$

for all $U, V, W \in \mathrm{Ob}\,\mathcal{C}$.

The MacLane Coherence Theorem [**18**, §VII.2] states that the pentagon and triangle axioms ensure that any for any two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by inserting $\mathbf{1}$'s and parentheses, all isomorphisms of these two expressions consisting of compositions of α's, λ's and ρ's are equal. This condition is called the *associativity axiom*. In a monoidal category, we can use the natural isomorphisms to identify all expressions of the above type and so we often write multiple tensor products without brackets. In fact, every monoidal category is equivalent to a strict one [**18**, §XI.3].

2.2. Examples. Most of the familiar tensor products yield monoidal categories. For instance, for a commutative ring R, the category of R-modules is a monoidal category. We have the usual tensor product $A \otimes_R B$ of modules A and B. The unit object is R and we have the natural isomorphisms

$$\alpha : A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C, \quad \alpha(a \otimes_R (b \otimes_R c)) = (a \otimes_R b) \otimes_R c$$

$$\lambda : R \otimes_R A \cong A, \quad \lambda(r \otimes_R a) = ra,$$

$$\rho : A \otimes_R R \cong A, \quad \rho(a \otimes_R r) = ra.$$

In particular, the categories of abelian groups (where $R = \mathbb{Z}$) and vector spaces (where R is a field) are monoidal categories. In a similar fashion, the category

of R-algebras is monoidal under the usual tensor product of algebras. For an arbitrary (not necessarily commutative) ring R, the category of R-R bimodules is also monoidal under $\otimes_R$. The categories of sets and topological spaces with the cartesian product are monoidal categories with unit objects 1 (the set with a single element) and $*$ (the single-element topological space) respectively.

3. Braided monoidal categories

3.1. Definitions.

DEFINITION 3.1 (Braided monoidal category). *A* braided monoidal category *(or* braided tensor category*) is a monoidal category* $\mathcal{C}$ *equipped with natural isomorphisms* $\sigma_{U,V} : U \otimes V \to V \otimes U$ *for all* $U,V \in \mathrm{Ob}\,\mathcal{C}$ *satisfying the* hexagon axiom: *for all* $U,V,W \in \mathcal{C}$, *the diagrams*

$$
\begin{array}{ccc}
& U \otimes (V \otimes W) \xrightarrow{\sigma_{U,V\otimes W}} (V \otimes W) \otimes U & \\
\alpha_{U,V,W} \nearrow & & \searrow \alpha_{V,W,U} \\
(U \otimes V) \otimes W & & V \otimes (W \otimes U) \\
\searrow \sigma_{U,V}\otimes\mathrm{id}_W & & \nearrow \mathrm{id}_V\otimes\sigma_{U,W} \\
& (V \otimes U) \otimes W \xrightarrow{\alpha_{V,U,W}} V \otimes (U \otimes W) &
\end{array}
$$

and

$$
\begin{array}{ccc}
& U \otimes (V \otimes W) \xrightarrow{\sigma^{-1}_{V\otimes W,U}} (V \otimes W) \otimes U & \\
\alpha_{U,V,W} \nearrow & & \searrow \alpha_{V,W,U} \\
(U \otimes V) \otimes W & & V \otimes (W \otimes U) \\
\searrow \sigma^{-1}_{V,U}\otimes\mathrm{id}_W & & \nearrow \mathrm{id}_V\otimes\sigma^{-1}_{W,U} \\
& (V \otimes U) \otimes W \xrightarrow{\alpha_{V,U,W}} V \otimes (U \otimes W) &
\end{array}
$$

commute. The collection of maps $\sigma_{U,V}$ *is called a* braiding.

For a braided monoidal category $\mathcal{C}$ and $U,V,W \in \mathrm{Ob}\,\mathcal{C}$, consider the following diagram where we have omitted bracketings and associators (or assumed that $\mathcal{C}$ is strict).

$$
\begin{array}{ccc}
& V \otimes U \otimes W & \\
\sigma_{U,V}\otimes\mathrm{id}_W \nearrow & & \searrow \mathrm{id}_V\otimes\sigma_{U,W} \\
U \otimes V \otimes W \xrightarrow{\sigma_{U,V\otimes W}} & & V \otimes W \otimes U \\
\mathrm{id}_U\otimes\sigma_{V,W} \downarrow & & \downarrow \sigma_{V,W}\otimes\mathrm{id}_U \\
U \otimes W \otimes V \xrightarrow{\sigma_{U,W\otimes V}} & & W \otimes V \otimes U \\
\sigma_{U,W}\otimes\mathrm{id}_V \searrow & & \nearrow \mathrm{id}_W\otimes\sigma_{U,V} \\
& W \otimes U \otimes V &
\end{array}
$$

The top and bottom triangles commute by the hexagon axiom and the middle rectangle commutes by the naturality of the braiding (that is, by the fact that it

is a natural isomorphism). Therefore, if we write σ_1 for the map $\sigma \otimes \mathrm{id}$ and σ_2 for the map $\mathrm{id} \otimes \sigma$, we have

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2,$$

the Yang-Baxter relation for $\mathcal{B}_3$. It follows that in a braided monoidal category, the braid group $\mathcal{B}_n$ acts on n-fold tensor products. That is, if we denote $\mathrm{id}^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{id}^{\otimes(n-i-1)}$ by σ_i, then a composition of such maps depends only on the corresponding element of the braid group.

DEFINITION 3.2 (Symmetric monoidal category). *A symmetric monoidal category* $\mathcal{C}$ *where* $\sigma_{V,U} \circ \sigma_{U,V} = \mathrm{id}_{U \otimes V}$ *for all* $U, V \in$ $\mathrm{Ob}\,\mathcal{C}$.

In any symmetric monoidal category, the symmetric group S_n acts on n-fold tensor products in the same way that the braid group acts in a braided monoidal category. We note that there is conflicting terminology in the literature. For instance, some authors refer to monoidal categories as tensor categories while others (see, for instance, [**2**]) refer to symmetric monoidal categories as tensor categories and braided monoidal categories as quasitensor categories.

3.2. Examples. Many of the examples in Section 2.2 can in fact be given the structure of a symmetric monoidal category. In particular, the categories of R-modules and R-algebras over a commutative ring R, the category of sets, and the category of topological spaces are all symmetric monoidal categories. In all of these examples, the braiding is given by $\sigma(a \otimes b) = b \otimes a$.

An example that shall be especially important to us is the category of representations of a quantum group. It can be given the structure of a braided monoidal category but is not a symmetric monoidal category.

4. Coboundary monoidal categories

4.1. Definitions. Coboundary monoidal categories are analogues of braided monoidal categories in which the role of the braid group is replaced by the cactus group. As we shall see, they are better suited to the theory of crystals than braided monoidal categories.

DEFINITION 4.1 (Coboundary monoidal category). *A coboundary monoidal category is a monoidal category* $\mathcal{C}$ *together with natural isomorphisms* $\sigma_{U,V}^c : U \otimes V \to V \otimes U$ *for all* $U, V \in \mathrm{Ob}\,\mathcal{C}$ *satisfying the following conditions:*

(1) $\sigma_{V,U}^c \circ \sigma_{U,V}^c = \mathrm{id}_{U \otimes V}$, *and*
(2) *the cactus relation: for all* $U, V, W \in \mathrm{Ob}\,\mathcal{C}$, *the diagram*

(4.1)
$$
\begin{array}{ccc}
U \otimes V \otimes W & \xrightarrow{\sigma_{U,V}^c \otimes \mathrm{id}_W} & V \otimes U \otimes W \\
{\scriptstyle \mathrm{id}_U \otimes \sigma_{V,W}^c} \downarrow & & \downarrow {\scriptstyle \sigma_{V \otimes U, W}^c} \\
U \otimes W \otimes V & \xrightarrow{\sigma_{U, W \otimes V}^c} & W \otimes V \otimes U
\end{array}
$$

commutes.

The collection of maps $\sigma_{U,V}^c$ *is called a* cactus commutor.

We will use the term *commutor* for a collection of natural isomorphisms $\sigma_{U,V} :$ $U \otimes V \to V \otimes U$ for all objects $U, V \in \mathrm{Ob}\,\mathcal{C}$ (this is sometimes called a *commutativity*

constraint), reserving the term *cactus commutor* for a commutor satisfying the conditions in the definition above.

Suppose $\mathcal{C}$ is a coboundary category and $U_1, \ldots, U_n \in \mathrm{Ob}\,\mathcal{C}$. For $1 \le p < q \le n$, one defines natural isomorphisms

$$\sigma_{p,q}^c = (\sigma_{p,q}^c)_{U_1,\ldots,U_n} : U_1 \otimes \cdots \otimes U_n$$
$$\to U_1 \otimes \cdots \otimes U_{p-1} \otimes U_{p+1} \otimes \cdots \otimes U_q \otimes U_p \otimes U_{q+1} \otimes \cdots \otimes U_n,$$

$$\sigma_{p,q}^c := \mathrm{id}_{U_1 \otimes \cdots \otimes U_{p-1}} \otimes \sigma_{U_p, U_{p+1} \otimes \cdots \otimes U_q}^c \otimes \mathrm{id}_{U_{q+1} \otimes \cdots \otimes U_n}.$$

For $1 \le p < q \le n$, one then defines natural isomorphisms

$$s_{p,q} : U_1 \otimes \cdots \otimes U_n \to U_1 \otimes U_2 \otimes \cdots \otimes U_{p-1} \otimes U_q \otimes U_{q-1} \otimes \ldots U_p \otimes U_{q+1} \otimes U_{q+2} \otimes \cdots \otimes U_n$$

recursively as follows. Define $s_{p,p+1} = \sigma_{p,p+1}^c$ and $s_{p,q} = \sigma_{p,q}^c \circ s_{p+1,q}$ for $q > p + 1$. We also set $s_{p,p} = \mathrm{id}$. The following proposition was proved by Henriques and Kamnitzer.

PROPOSITION 4.2 ([**7**, Lemma 3, Lemma 4]). *If $\mathcal{C}$ is a coboundary category and the natural isomorphisms $s_{p,q}$ are defined as above, then*

(1) $s_{p,q} \circ s_{p,q} = \mathrm{id}$,
(2) $s_{p,q} \circ s_{k,l} = s_{k,l} \circ s_{p,q}$ *if $p < q$ and $k < l$ are disjoint, and*
(3) $s_{p,q} \circ s_{k,l} = s_{r,t} \circ s_{p,q}$ *if $p < q$ contains $k < l$, where $r = \hat{s}_{p,q}(l)$ and $t = \hat{s}_{p,q}(k)$.*

Therefore, in a coboundary monoidal category the cactus group J_n acts on n-fold tensor products. This is analogous to the action of the braid group in a braided monoidal category. For this reason, the authors of [**7**] propose the name *cactus category* for a coboundary monoidal category.

PROPOSITION 4.3. *Any braiding satisfies the cactus relation.*

PROOF. Suppose $\mathcal{C}$ is a braided monoidal category with braiding σ and $U, V, W \in \mathrm{Ob}\,\mathcal{C}$. By the axioms of a braided monoidal category, we have

$$\sigma_{V \otimes U, W}^{-1} = (\mathrm{id}_V \otimes \sigma_{U,W}^{-1})(\sigma_{V,W}^{-1} \otimes \mathrm{id}_U)$$

Taking inverses gives

$$\sigma_{V \otimes U, W} = (\sigma_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes \sigma_{U,W}).$$

Therefore

$$\begin{aligned}
(\sigma_{U, W \otimes V})(\mathrm{id}_U \otimes \sigma_{V,W}) &= (\mathrm{id}_W \otimes \sigma_{U,V})(\sigma_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \sigma_{V,W}) \\
&= (\sigma_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes \sigma_{U,W})(\sigma_{U,V} \otimes \mathrm{id}_W) \\
&= (\sigma_{V \otimes U, W})(\sigma_{U,V} \otimes \mathrm{id}_W).
\end{aligned}$$

The first equality uses the definition of a braiding and the second is the Yang-Baxter equation.   □

Note that Proposition 4.3 does not imply that every braided monoidal category is a coboundary monoidal category because we require cactus commutors to be involutions whereas braidings, in general, are not. In fact, any commutor that is both a braiding and a cactus commutor is in fact a symmetric commutor (that is, it endows the category in question with the structure of a symmetric monoidal category).

4.2. Examples. The definition of coboundary monoidal categories was first given by Drinfel′d in [**5**]. The name was inspired by the fact that the representation categories of coboundary Hopf algebras are coboundary monoidal categories. Since the cactus group surjects onto the symmetric group, any symmetric monoidal category is a coboundary monoidal category. Our main example of coboundary categories which are not symmetric monoidal categories will be the categories of representations of quantum groups and crystals. Furthermore, we will see that the category of crystals cannot be given the structure of a braided monoidal category. Thus there exist examples of coboundary monoidal categories that are not braided.

5. Quantum groups and crystals

5.1. Quantum groups. Compact groups and semisimple Lie algebras are rigid objects in the sense that they cannot be deformed. However, if one considers the group algebra or universal enveloping algebra instead, a deformation is possible. Such a deformation can be carried out in the category of (noncommutative, non-cocommutative) Hopf algebras. These deformations play an important role in the study of the quantum Yang-Baxter equation and the quantum inverse scattering method. Another benefit is that the structure of the deformations and their representations becomes more rigid and the concepts of canonical bases and crystals emerge.

We introduce here the quantum group, or quantized enveloping algebra, defined by Drinfel′d and Jimbo. For further details we refer the reader to the many books on the subject (e.g. [**2, 8, 17**]). Let $\mathfrak{g}$ be a Kac-Moody algebra with symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ and symmetrizing matrix $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$. Let $P, P^\vee$, and Q_+ be the weight lattice, coweight lattice and positive root lattice respectively. Let $\mathbb{C}_q$ be the field $\mathbb{C}(q^{1/2})$ where q is a formal variable. For $n \in \mathbb{Z}$ and any symbol x, we define

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad [0]_x! = 1, \quad [m]_x! = [m]_x[m-1]_x \cdots [1]_x \text{ for } m \in \mathbb{Z}_{>0},$$

$$\begin{bmatrix} k \\ l \end{bmatrix}_x = \frac{[k]_x!}{[l]_x![k-l]_x!} \quad \text{for } k, l \in \mathbb{Z}_{\geq 0}.$$

DEFINITION 5.1 (Quantum group $U_q(\mathfrak{g})$). *The* quantum group *or* quantized enveloping algebra $U_q(\mathfrak{g})$ *is the unital associative algebra over* $\mathbb{C}_q$ *with generators* e_i, f_i $(i \in I)$ *and* q^h $(h \in P^\vee)$ *with defining relations*

(1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ *for* $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ *for* $h \in P^\vee$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ *for* $h \in P^\vee$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{q^{s_i h_i} - q^{-s_i h_i}}{q^{s_i} - q^{-s_i}}$ *for* $i, j \in I$,

(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$,

(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$.

As $q \to 1$, the defining relations for $U_q(\mathfrak{g})$ approach the usual relations for $\mathfrak{g}$ in the following sense. Taking the "derivative" with respect to q in the second and

third relations gives

$$hq^{h-1}e_iq^{-h} + q^he_i\left(-hq^{-h-1}\right) = \alpha_i(h)q^{\alpha_i(h)-1}e_i \xrightarrow{q\to 1} he_i - e_ih = [h, e_i] = \alpha_i(h)e_i,$$

$$hq^{h-1}f_iq^{-h} + q^hf_i\left(-hq^{-h-1}\right) = -\alpha_i(h)q^{-\alpha_i(h)-1}f_i \xrightarrow{q\to 1} hf_i - f_ih = [h, f_i] = -\alpha_i(h)f,$$

Furthermore, if we naively apply L'Hôpital's rule, we have

$$\lim_{q\to 1}\frac{q^{s_ih_i} - q^{-s_ih_i}}{q^{s_i} - q^{-s_i}} = \lim_{q\to 1}\frac{s_ih_iq^{s_ih_i-1} + s_ih_iq^{-s_ih_i-1}}{s_iq^{s_i-1} + s_iq^{-s_i-1}} = \frac{2s_ih_i}{2s_i} = h_i,$$

and so the fourth defining relation of $U_q(\mathfrak{g})$ becomes $[e_i, f_j] = \delta_{ij}h_i$ in the $q \to 1$ limit – called the *classical limit*. Similarly, since we have

$$[n]_{q^{s_i}} \to n, \text{ and } \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{s_i}} \to \binom{1 - a_{ij}}{k} \text{ as } q \to \infty,$$

the last two relations (called the *quantum Serre relations*) become the usual Serre relations in the classical limit. Thus, we can think of $U_q(\mathfrak{g})$ as a deformation of $\mathfrak{g}$. For a more rigorous treatment of the classical limit, we refer the reader to [**8**, §3.4].

The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure given by comultiplication

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes q^{s_ih_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{-s_ih_i} \otimes f_i,$$

counit

$$\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

and antipode

$$\gamma(q^h) = q^{-h}, \quad \gamma(e_i) = -e_iq^{-s_ih_i}, \quad \gamma(f_i) = -q^{s_ih_i}f_i.$$

There are other choices but we will use the above in what follows.

The representations of $U(\mathfrak{g})$ can be q-deformed to representations of $U_q(\mathfrak{g})$ in such a way that the dimensions of the weight spaces are invariant under the deformation (see [**8**, **15**]). The q-deformed notion of a weight space is as follows: for a $U_q(\mathfrak{g})$-module M and $\lambda \in P$, the λ-weight space of M is

$$M^\lambda = \{v \in M \mid q^hv = q^{\lambda(h)}v \,\forall\, h \in P^\vee\}.$$

5.2. Crystal bases. In this section we introduce the theory of crystal bases, which can be thought of as the $q \to \infty$ limit of the representation theory of quantum groups. In this limit, representations are replaced by combinatorial objects called crystal graphs. These objects, which are often much easier to compute with than the representations themselves, can be used to obtain such information as dimensions of weight spaces (characters) and the decomposition of tensor products into sums of irreducible representations. For details, we refer the reader to [**8**]. We note that in [**8**], the limit $q \to 0$ is used. This simply corresponds to a different choice of Hopf algebra structure on $U_q(\mathfrak{g})$. We choose to consider $q \to \infty$ to match the choices of [**2**].

For $n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, define the *divided powers*

$$e_i^{(n)} = e_i^n/[n]_q!, \quad f_i^{(n)} = f_i^n/[n]_q!$$

Let M be an integrable $U_q(\mathfrak{g})$-module and let M^λ be the λ-weight space for $\lambda \in P$. For $i \in I$, any weight vector $u \in M^\lambda$ can be written uniquely in the form

$$u = \sum_{n=0}^\infty f_i^{(n)}u_n, \quad u_n \in \ker e_i \cap M^{\lambda+n\alpha_i}.$$

Define the *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i : M \to M$ by

$$\tilde{e}_i u = \sum_{n=1}^{\infty} f_i^{(n-1)} u_n, \quad \tilde{f}_i u = \sum_{n=0}^{\infty} f_i^{(n+1)} u_n.$$

Let A be the integral domain of all rational functions in $\mathbb{C}_q$ that are regular at $q = \infty$. That is, A consists of all rational functions that can be written in the form $g_1(q^{-1/2})/g_2(q^{-1/2})$ for $g_1(q^{-1/2})$ and $g_2(q^{-1/2})$ polynomials in $\mathbb{C}[q^{-1/2}]$ with $g_2(q^{-1/2})|_{q^{-1/2}=0} \neq 0$ (one should think of these as rational functions whose limit exists as $q \to \infty$).

DEFINITION 5.2 (Crystal basis). *A crystal basis of a $U_q(\mathfrak{g})$-module M is a pair (L, B) such that*

(1) *L is a free A-submodule of M such that $M = \mathbb{C}_q \otimes_A L$,*
(2) *B is a $\mathbb{C}$-basis of the vector space $L/q^{-1/2}L$ over $\mathbb{C}$,*
(3) *$L = \bigoplus_\lambda L^\lambda$, $B = \bigsqcup_\lambda B^\lambda$ where $L^\lambda = L \cap M^\lambda$, $B^\lambda = B \cap L^\lambda/q^{-1/2}L^\lambda$,*
(4) *$\tilde{e}_i L \subseteq L$, $\tilde{f}_i L \subseteq L$ for all $i \in I$,*
(5) *$\tilde{e}_i B \subseteq B \cup \{0\}$, $\tilde{f}_i B \subseteq B \cup \{0\}$ for all $i \in I$, and*
(6) *for all $b, b' \in B$ and $i \in I$, $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$.*

It was shown by Kashiwara [11] that all $U_q(\mathfrak{g})$-modules in the category $\mathcal{O}_{\text{int}}^q$ (integrable modules with weight space decompositions and weights lying in a union of sets of the form $\lambda - Q_+$ for $\lambda \in P$) have unique crystal bases (up to isomorphism).

A crystal basis can be represented by a *crystal graph*. The crystal graph corresponding to a crystal basis (L, B) is an edge-colored (by I) directed graph with vertex set B and a i-colored directed edge from b' to b if $\tilde{f}_i b' = b$ (equivalently, if $\tilde{e}_i b = b'$). Crystals can be defined in a more abstract setting where a crystal consists of such a graph along with maps wt $: B \to P$ and $\varphi_i, \varepsilon_i : B \to \mathbb{Z}_{\geq 0}$ satisfying certain axioms. In this paper, by the *category of $\mathfrak{g}$-crystals* for a symmetrizable Kac-Moody algebra $\mathfrak{g}$, we mean the category consisting of those crystal graphs B such that each connected component of B is isomorphic to some B_λ, the crystal corresponding to the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ, where λ is a dominant integral weight. In this case

$$\text{wt}(b) = \mu \text{ for } b \in B^\mu, \quad \varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq 0\}, \quad \varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k \neq 0\}.$$

For the rest of this paper, the word $\mathfrak{g}$-*crystal* means an object in this category.

EXAMPLE 5.3 (Crystal bases of finite-dimensional representations of $U_q(\mathfrak{sl}_2)$). *For $n \in \mathbb{Z}_{\geq 0}$, let V_n be the irreducible $U_q(\mathfrak{sl}_2)$-module of highest weight n. It is a q-deformation of the corresponding $\mathfrak{sl}_2$-module. Let v_n be a highest weight vector of V_n and define*

$$v_{n-2i} = f^{(i)} v_n.$$

Then $\{v_n, v_{n-2}, \ldots, v_{-n}\}$ is a basis of V_n and v_j has weight j. Let

$$L = \text{Span}_A\{v_n, v_{n-2}, \ldots, v_{-n}\}, \quad \text{and}$$
$$B = \{b_n, b_{n-2}, \ldots, b_{-n}\}$$

where b_j is the image of v_j in the quotient $L/q^{-1/2}L$. It is easily checked that (L, B) is a crystal basis of V_n and the corresponding crystal graph is

$$b_n \longrightarrow b_{n-2} \longrightarrow \cdots \longrightarrow b_{-n}$$

(since there is only one simple root for $\mathfrak{sl}_2$, we omit the edge-coloring).

5.3. Tensor products. One of the nicest features of the theory of crystals is the existence of the *tensor product rule* which tells us how to form the crystal corresponding to the tensor product of two representations from the crystals corresponding to the two factors.

THEOREM 5.4 (Tensor product rule [**8**, Theorem 4.4.1], [**2**, Proposition 14.1.14]). *Suppose (L_j, B_j) are crystal bases of $U_q(\mathfrak{g})$-modules M_j $(j = 1, 2)$ in $\mathcal{O}_{int}^q$. For $b \in B_j$ and $i \in I$, let*

$$\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq 0\}, \quad \varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq 0\}.$$

Let $L = L_1 \otimes_A L_2$ and $B = B_1 \times B_2$. Then (L, B) is a crystal basis of $M_1 \otimes_{\mathbb{C}_q} M_2$, where the action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ are given by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Here we write $b_1 \otimes b_2$ for $(b_1, b_2) \in B_1 \times B_2$ and $b_1 \times 0 = 0 \times b_2 = 0$.

We write $B_1 \otimes B_2$ for the crystal graph $B_1 \times B_2$ of $M_1 \otimes M_2$ with crystal operators defined by the formulas in Theorem 5.4. Note that even though we use a different coproduct than in [**8**], the tensor product rule remains the same as seen in [**2**, Proposition 14.1.14].

5.4. The braiding in the quantum group. The category of representations of $U(\mathfrak{g})$, the universal enveloping algebra of a symmetrizable Kac-Moody algebra $\mathfrak{g}$, is a symmetric monoidal category with braiding given by

$$\sigma_{U,V} : U \otimes V \to V \otimes U, \quad \sigma(u \otimes v) = \text{flip}(u \otimes v) \stackrel{\text{def}}{=} v \otimes u \quad \text{for} \quad u \in U, \ v \in V.$$

However, the analogous map is not a morphism in the category of representations of $U_q(\mathfrak{g})$ and this category is not a symmetric monoidal category. However, it is a braided monoidal category with a braiding constructed as follows. The *R-matrix* is an invertible element in a certain completed tensor product $U_q(\mathfrak{g}) \widehat{\otimes} U_q(\mathfrak{g})$ (see [**2, 10**]). It defines a map $U \otimes V \to U \otimes V$ for any representations U and V of $U_q(\mathfrak{g})$. The map given by

$$\sigma_{U,V} : U \otimes V \to V \otimes U, \quad \sigma_{U,V} = \text{flip} \circ R$$

for representations U and V of $U_q(\mathfrak{g})$ is a braiding.

As an example, consider the representation $V_1 \otimes V_1$ of $U_q(\mathfrak{sl}_2)$. In the basis $S_1 = \{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$, the R-matrix is given by (see [**2**, Example 6.4.12])

$$R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Note that $R|_{q=1} = \mathrm{id}$ and so in the classical limit, the braiding becomes the map flip. In the basis S_1, we have

$$(5.1) \quad \mathrm{flip} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and so} \quad \mathrm{flip} \circ R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Now consider the basis

$$S_2 = \{v_1 \otimes v_1, a, b, v_{-1} \otimes v_{-1}\}, \quad a = v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}, \quad b = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}.$$

Note that

$$ea = fa = 0, \quad f(v_1 \otimes v_1) = b.$$

Thus S_2 is a basis of $V_1 \otimes V_1$ compatible with the decomposition $V_1 \otimes V_1 \cong V_0 \oplus V_2$. In the basis S_2, we have

$$(5.2) \qquad \mathrm{flip} \circ R = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & -q^{-3/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}.$$

From this form, we easily see that $\mathrm{flip} \circ R$ is an isomorphism of $U_q(\mathfrak{sl}_2)$-modules $V_1 \otimes V_1 \to V_1 \otimes V_1$. It acts as multiplication by $q^{1/2}$ on the summand V_2 and by $-q^{-3/2}$ on the summand V_0.

Now, from Example 5.3 and the tensor product rule (Theorem 5.4), we see that the crystal basis of $V_1 \otimes V_1$ is given by

$$L = \mathrm{Span}_A\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}, \quad \text{and}$$
$$B = \{b_1 \otimes b_1, b_{-1} \otimes b_1, b_1 \times b_{-1}, b_{-1} \otimes b_{-1}\}.$$

From the matrix of $\mathrm{flip} \circ R$ in the basis S_1 given in (5.1), we see that it does not preserve the crystal lattice L since it involves positive powers of q. Furthermore, there is no $\mathbb{C}_q$-multiple of $\mathrm{flip} \circ R$ which preserves L and induces an isomorphism of $L/q^{-1/2}L$. To see this, note from (5.1) that in order for $g(q) \, \mathrm{flip} \circ R$, with $g(q) \in \mathbb{C}_q$, to preserve the crystal lattice, we would need $q^{1/2}g(q) \in A$ and thus $q^{-1/2}g(q) \in q^{-1}A \subseteq q^{-1/2}A$. However, we would then have

$$g(q) \, \mathrm{flip} \circ R(v_1 \otimes v_{-1}) = q^{-1/2}g(q)v_{-1} \otimes v_1 \equiv 0 \mod q^{-1/2}L$$

and so $g(q) \, \mathrm{flip} \circ R$ would not induce an isomorphism of $L/q^{-1/2}L$. Therefore, we see that the braiding coming from the R-matrix does not pass to the $q \to \infty$ limit. That is, it does not induce a braiding on the crystal $B_1 \otimes B_1$.

It turns out that the above phenomenon is unavoidable. That is, no braiding on the category of representations of a quantum group passes to the $q \to \infty$ limit. In fact, we have the following even stronger results.

LEMMA 5.5. *The category of $\mathfrak{sl}_2$-crystals cannot be given the structure of a braided monoidal category.*

PROOF. We prove the result by contradiction. Suppose the category of $\mathfrak{sl}_2$-crystals is a braided monoidal category with braiding σ. Consider the crystal B_1. It has crystal graph

$$B_1 \; : \; b_1 \longrightarrow b_{-1}.$$

The crystal graph of the tensor product $B_1 \otimes B_1$ has two connected components:

$$b_1 \otimes b_1 \longrightarrow b_{-1} \otimes b_1 \longrightarrow b_{-1} \otimes b_{-1} \quad \cong B_2$$
$$b_1 \otimes b_{-1} \quad \cong B_0$$

Since σ_{B_1,B_1} is an crystal isomorphism, we see from the above that it must act as the identity. Therefore

$$(5.3) \qquad (\mathrm{id}_{B_1} \otimes \sigma_{B_1,B_1}) \circ (\sigma_{B_1,B_1} \otimes \mathrm{id}_{B_1}) = \mathrm{id}_{B_1 \otimes B_1 \otimes B_1}.$$

Now, the graph of the crystal $B_1 \otimes B_2$ has two connected components:

$$b_1 \otimes b_2 \longrightarrow b_{-1} \otimes b_2 \longrightarrow b_{-1} \otimes b_0 \longrightarrow b_{-1} \otimes b_{-2} \quad \cong B_3$$
$$b_1 \otimes b_0 \longrightarrow b_1 \otimes b_{-2} \quad \cong B_1$$

The graph of the crystal $B_2 \otimes B_1$ also has two connected components:

$$b_2 \otimes b_1 \longrightarrow b_0 \otimes b_1 \longrightarrow b_{-2} \otimes b_1 \longrightarrow b_{-2} \otimes b_{-1} \quad \cong B_3$$
$$b_2 \otimes b_{-1} \longrightarrow b_0 \otimes b_{-1} \quad \cong B_1$$

Since σ_{B_1,B_2} is a crystal isomorphism, we must have $\sigma_{B_1,B_2}(b_1 \otimes b_0) = b_2 \otimes b_{-1}$.

Now, consider the inclusion of crystals $j : B_2 \hookrightarrow B_1 \otimes B_1$ given by

$$j(b_2) = b_1 \otimes b_1, \quad j(b_0) = b_{-1} \otimes b_1, \quad j(b_{-2}) = b_{-1} \otimes b_{-1}.$$

By the naturality of the braiding, the following diagram commutes:

$$
\begin{array}{ccc}
B_1 \otimes B_2 & \xrightarrow{\;\mathrm{id}_{B_1} \otimes j\;} & B_1 \otimes B_1 \otimes B_1 \\
\downarrow{\scriptstyle \sigma_{B_1,B_2}} & & \downarrow{\scriptstyle \sigma_{B_1,B_1 \otimes B_1}} \\
B_2 \otimes B_1 & \xrightarrow{\;j \otimes \mathrm{id}_{B_1}\;} & B_1 \otimes B_1 \otimes B_1
\end{array}
$$

We therefore have

$$\sigma_{B_1,B_1 \otimes B_1}(b_1 \otimes b_{-1} \otimes b_1) = \sigma_{B_1,B_1 \otimes B_1} \circ (\mathrm{id}_{B_1} \otimes j)(b_1 \otimes b_0) = (j \otimes \mathrm{id}_{B_1}) \circ \sigma_{B_1,B_2}(b_1 \otimes b_0)$$
$$= (j \otimes \mathrm{id}_{B_1})(b_2 \otimes b_{-1}) = b_1 \otimes b_1 \otimes b_{-1}.$$

Comparing to (5.3), we see that

$$\sigma_{B_1,B_1 \otimes B_1} \neq (\mathrm{id}_{B_1} \otimes \sigma_{B_1,B_1}) \circ (\sigma_{B_1,B_1} \otimes \mathrm{id}_{B_1}),$$

contradicting the fact that σ is a braiding. $\square$

An alternative proof of Lemma 5.5 was given in [**7**]. We can use the fact that $\mathfrak{g}$-crystals, for $\mathfrak{g}$ a symmetrizable Kac-Moody algebra, can be restricted to $\mathfrak{sl}_2$-crystals to generalize this result.

PROPOSITION 5.6. *For any symmetrizable Kac-Moody algebra* $\mathfrak{g}$*, the category of* $\mathfrak{g}$*-crystals cannot be given the structure of a braided monoidal category.*

PROOF. We prove the result by contradiction. Suppose the category of $\mathfrak{g}$-crystals was a braided monoidal category with braiding σ for some symmetrizable Kac-Moody algebra $\mathfrak{g}$. Let α_1 and ω_1 be a simple root and fundamental weight (respectively) corresponding to some vertex in the Dynkin diagram of $\mathfrak{g}$. The restriction of a $\mathfrak{g}$-crystal to the color 1 yields an $\mathfrak{sl}_2$-crystal. More precisely, one forgets the operators $\tilde{e}_i$, $\tilde{f}_i$, φ_i and ε_i for $i \neq 1$ and projects the map wt to the one-dimensional sublattice $\mathbb{Z}\omega_1 \subseteq P$. In general, even if the original $\mathfrak{g}$-crystal was

connected (i.e. irreducible), the induced $\mathfrak{sl}_2$-crystal will not be. However, any morphism of $\mathfrak{g}$-crystals induces a morphism of the restricted $\mathfrak{sl}_2$-crystals.

Consider the $\mathfrak{g}$-crystal $B_{k\omega_1}$, $k \geq 1$, corresponding to the irreducible $U_q(\mathfrak{g})$-module of highest weight $k\omega_1$. If we restrict this to an $\mathfrak{sl}_2$-crystal, the connected component of the $\mathfrak{sl}_2$-crystal graph containing the highest weight element $b_{k\omega_1}$ is isomorphic to the $\mathfrak{sl}_2$-crystal B_k. Now, since $b_{\omega_1} \otimes b_{\omega_1}$ is the unique element of $B_{\omega_1} \otimes B_{\omega_1}$ of weight $2\omega_1$, we have

$$\sigma_{B_{\omega_1}, B_{\omega_1}}(b_{\omega_1} \otimes b_{\omega_1}) = b_{\omega_1} \otimes b_{\omega_1}.$$

The connected $\mathfrak{sl}_2$-subcrystal containing the element $b_{\omega_1} \otimes b_{\omega_1}$ is

$$(5.4) \qquad b_{\omega_1} \otimes b_{\omega_1} \longrightarrow \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1} \longrightarrow \tilde{f}_1 b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \quad \cong B_2$$

(as in the proof of Lemma 5.5). Thus, since $\sigma_{B_{\omega_1} \otimes B_{\omega_1}}$ is a morphism of $\mathfrak{sl}_2$-crystals, we must have

$$\sigma_{B_{\omega_1} \otimes B_{\omega_1}}(\tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}) = \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}.$$

Now, the only element of $B_{\omega_1} \otimes B_{\omega_1}$ of weight $2\omega_1 - \alpha_1$ not contained in the connected $\mathfrak{sl}_2$-subcrystal mentioned above is $b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1}$. Therefore, we must also have

$$\sigma_{B_{\omega_1}, B_{\omega_1}}(b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1}) = b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1}.$$

Thus,

$$(5.5) \quad (\mathrm{id}_{B_{\omega_1}} \otimes \sigma_{B_{\omega_1}, B_{\omega_1}}) \circ (\sigma_{B_{\omega_1}, B_{\omega_1}} \otimes \mathrm{id}_{B_{\omega_1}})(b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1})$$
$$= (\mathrm{id}_{B_{\omega_1}} \otimes \sigma_{B_{\omega_1}, B_{\omega_1}})(b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}) = b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}.$$

Now, the connected $\mathfrak{sl}_2$-subcrystal of $B_{\omega_1} \otimes B_{2\omega_1}$ containing the element $b_{\omega_1} \otimes b_{2\omega_1}$ is

$$b_{\omega_1} \otimes b_{2\omega_1} \longrightarrow \tilde{f}_1 b_{\omega_1} \otimes b_{2\omega_1} \longrightarrow \tilde{f}_1 b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1} \longrightarrow \tilde{f}_1 b_{\omega_1} \otimes \tilde{f}_1^2 b_{2\omega_1} \quad \cong B_3,$$

and the connected $\mathfrak{sl}_2$-subcrystal containing the element $b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1}$ is

$$b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1} \longrightarrow b_{\omega_1} \otimes \tilde{f}_1^2 b_{2\omega_1} \quad \cong B_1$$

(as in the proof of Lemma 5.5). Similarly, we have the following $\mathfrak{sl}_2$-subcrystals of $B_{2\omega_1} \otimes B_{\omega_1}$:

$$b_{2\omega_1} \otimes b_{\omega_1} \longrightarrow \tilde{f}_1 b_{2\omega_1} \otimes b_{\omega_1} \longrightarrow \tilde{f}_1^2 b_{2\omega_1} \otimes b_{\omega_1} \longrightarrow \tilde{f}_1^2 b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \quad \cong B_3,$$
$$b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \longrightarrow \tilde{f}_1 b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \quad \cong B_1.$$

Now, since $b_{\omega_1} \otimes b_{2\omega_1}$ and $b_{2\omega_1} \otimes b_{\omega_1}$ are the unique elements of $B_{\omega_1} \otimes B_{2\omega_1}$ and $B_{2\omega_1} \otimes B_{\omega_1}$ (respectively) of weight $3\omega_1$, we must have

$$\sigma_{B_{\omega_1}, B_{2\omega_1}}(b_{\omega_1} \otimes b_{2\omega_1}) = b_{2\omega_1} \otimes b_{\omega_1}.$$

Also, since $b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1}$ and $b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1}$ are the only elements of $B_{\omega_1} \otimes B_{2\omega_1}$ and $B_{2\omega_1} \otimes B_{\omega_1}$ (respectively) of weight $3\omega_1 - \alpha_1$ not contained in the connected $\mathfrak{sl}_2$-subcrystal containing $b_{\omega_1} \otimes b_{2\omega_1}$ and $b_{2\omega_1} \otimes b_{\omega_1}$ (respectively), we must have

$$\sigma_{B_{\omega_1}, B_{2\omega_1}}(b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1}) = b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1}.$$

Now, consider the inclusion of $\mathfrak{g}$-crystals $j : B_{2\omega_1} \hookrightarrow B_{\omega_1} \otimes B_{\omega_1}$ determined by $j(b_{2\omega_1}) = b_{\omega_1} \otimes b_{\omega_1}$. Restricting to the connected $\mathfrak{sl}_2$-crystals containing the elements $b_{2\omega_1}$ and $b_{\omega_1} \otimes b_{\omega_1}$, we see from (5.4) that

$$j(\tilde{f}_1 b_{2\omega_1}) = \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}, \quad j(\tilde{f}_1^2 b_{2\omega_1}) = \tilde{f}_1 b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1}.$$

By the naturality of the braiding σ, the following diagram is commutative:

$$
\begin{array}{ccc}
B_{\omega_1} \otimes B_{2\omega_1} & \xrightarrow{\ \ \mathrm{id}_{B_{\omega_1}} \otimes j\ \ } & B_{\omega_1} \otimes B_{\omega_1} \otimes B_{\omega_1} \\[4pt]
\Big\downarrow{\scriptstyle \sigma_{B_{\omega_1}, B_{2\omega_1}}} & & \Big\downarrow{\scriptstyle \sigma_{B_{\omega_1}, B_{\omega_1} \otimes B_{\omega_1}}} \\[4pt]
B_{2\omega_1} \otimes B_{\omega_1} & \xrightarrow{\ \ j \otimes \mathrm{id}_{B_{\omega_1}}\ \ } & B_{\omega_1} \otimes B_{\omega_1} \otimes B_{\omega_1}
\end{array}
$$

Therefore

$$
\begin{aligned}
\sigma_{B_{\omega_1}, B_{\omega_1} \otimes B_{\omega_1}}(b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1} \otimes b_{\omega_1}) &= \sigma_{B_{\omega_1}, B_{\omega_1} \otimes B_{\omega_1}} \circ (\mathrm{id}_{B_{\omega_1}} \otimes j)(b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1}) \\
&= (j \otimes \mathrm{id}_{B_{\omega_1}}) \circ \sigma_{B_{\omega_1}, B_{2\omega_1}}(b_{\omega_1} \otimes \tilde{f}_1 b_{2\omega_1}) \\
&= (j \otimes \mathrm{id}_{B_{\omega_1}})(b_{2\omega_1} \otimes \tilde{f}_1 b_{\omega_1}) \\
&= b_{\omega_1} \otimes b_{\omega_1} \otimes \tilde{f}_1 b_{\omega_1}.
\end{aligned}
$$

Comparing this to (5.5) we see that

$$
\sigma_{B_{\omega_1}, B_{\omega_1} \otimes B_{\omega_1}} \neq (\mathrm{id}_{B_{\omega_1}} \otimes \sigma_{B_{\omega_1}, B_{\omega_1}}) \circ (\sigma_{B_{\omega_1}, B_{\omega_1}} \otimes \mathrm{id}_{B_{\omega_1}}).
$$

This contradicts the fact that σ is a braiding. $\square$

6. Crystals and coboundary categories

In this section, we discuss how the categories of $U_q(\mathfrak{g})$-modules and $\mathfrak{g}$-crystals can be given the structure of a coboundary category. For the case of $\mathfrak{g}$-crystals, we mention several different constructions and note the relationship between them.

6.1. Drinfel'd's unitarization. In [**5**], Drinfel'd defined the *unitarized R-matrix*

$$
\bar{R} = R(R^{\mathrm{op}}R)^{-1/2},
$$

where $R^{\mathrm{op}} = \mathrm{flip}(R)$ (as an operator on $M_1 \otimes M_2$, R^{op} acts as $\mathrm{flip} \circ R \circ \mathrm{flip}$) and the square root is taken with respect to a certain filtration on the completed tensor product $U_q(\mathfrak{g}) \hat{\otimes} U_q(\mathfrak{g})$. He then showed that $\mathrm{flip} \circ \bar{R}$ is a cactus commutor and so endows the category of $U_q(\mathfrak{g})$-modules with the structure of a coboundary category (see the comment after the proof of Proposition 3.3 in [**5**]). That is, it satisfies the conditions of Definition 4.1.

Consider the representation $V_1 \otimes V_1$ of $U_q(\mathfrak{sl}_2)$. In Section 5.4, we described the action of the R-matrix on this representation in two different bases S_1 and S_2. It follows from (5.2) that in the basis S_2,

$$
R^{\mathrm{op}}R = (\mathrm{flip} \circ R)^2 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-3} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.
$$

Therefore, we can take the (inverse of the) square root

$$
(R^{\mathrm{op}}R)^{-1/2} = q^{-1/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

In the basis S_1, we then have

$$(R^{\mathrm{op}}R)^{-1/2} = q^{-1/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2q^2}{1+q^2} & \frac{q-q^3}{1+q^2} & 0 \\ 0 & \frac{q-q^3}{1+q^2} & \frac{1+q^4}{1+q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{R} = R(R^{\mathrm{op}}R)^{-1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2q}{1+q^2} & \frac{1-q^2}{1+q^2} & 0 \\ 0 & \frac{q^2-1}{1+q^2} & \frac{2q}{1+q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, in the basis S_1,

$$(6.1) \quad \mathrm{flip} \circ R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \mathrm{flip} \circ \bar{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^2-1}{1+q^2} & \frac{2q}{1+q^2} & 0 \\ 0 & \frac{2q}{1+q^2} & \frac{1-q^2}{1+q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And in the basis S_2,

$$(6.2) \quad \mathrm{flip} \circ R = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & -q^{-3/2} & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}, \quad \mathrm{flip} \circ \bar{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the above, we have recalled the computation of $\mathrm{flip} \circ R$ from Section 5.4 for the purposes of comparison.

We note two important properties of $\mathrm{flip} \circ \bar{R}$. First of all, we see from (6.1) that the matrix coefficients in the basis S_1 of $\mathrm{flip} \circ \bar{R}$ lie in A, the ring of rational functions in $\mathbb{C}_q$ that are regular at $q = \infty$. Thus, $\mathrm{flip} \circ \bar{R}$ preserves the crystal lattice of $V_1 \otimes V_1$. In the $q \to \infty$ limit (more precisely, when passing to the quotient $L/q^{-1/2}L$), we have (in the basis S_1)

$$\mathrm{flip} \circ \bar{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathrm{mod}\ q^{-1/2}L.$$

Thus, $\mathrm{flip} \circ \bar{R}$ passes to the $q \to \infty$ limit and, up to signs, induces an involution on the crystal $B_1 \otimes B_1$ (see Section 6.4 for details). As noted in Section 5.4, the same is not true of $\mathrm{flip} \circ R$.

The second important property of $\mathrm{flip} \circ \bar{R}$ is that it is a cactus commutor. We see immediately from (6.2) that $(\mathrm{flip} \circ \bar{R})^2 = \mathrm{id}$. A straightforward (if somewhat lengthy) computation shows that $\mathrm{flip} \circ \bar{R}$ also satisfies the cactus relation (see [5, §3] for the proof in a more general setting).

The unitarized R-matrix has shown up in several different places. In [1] it arose naturally in the development of the theory of braided symmetric and exterior algebras. The reason for this is that if one wants to have interesting symmetric or exterior algebras, one needs an operator with eigenvalues of positive or negative one. Notice from (6.2) that while $\mathrm{flip} \circ R$ does not have this property, the operator $\mathrm{flip} \circ \bar{R}$ does. Essentially, in the basis S_2, the matrix for $\mathrm{flip} \circ \bar{R}$ is obtained from

the matrix for flip $\circ R$ by setting $q = 1$. Note that this does not imply that flip $\circ \bar{R}$ is an operator in the classical limit, merely that its matrix coefficients in a certain basis do not involve powers of q. The $q \to \infty$ limit of the unitarized R-matrix also appeared independently in the study of cactus commutors for crystals. We discuss this in the next two subsections.

6.2. The crystal commutor using the Schützenberger involution. Let $\mathfrak{g}$ be a simple complex Lie algebra and let I denote the set of vertices of the Dynkin graph of $\mathfrak{g}$. If w_0 is the long element in the Weyl group of $\mathfrak{g}$, let $\theta : I \to I$ be the involution such that $\alpha_{\theta(i)} = -w_0 \cdot \alpha_i$. Define a crystal $\overline{B_\lambda}$ with underlying set $\{\overline{b} \mid b \in B_\lambda\}$ and

$$\tilde{e}_i \cdot \overline{b} = \overline{\tilde{f}_{\theta(i)} \cdot b}, \quad \tilde{f}_i \cdot \overline{b} = \overline{\tilde{e}_{\theta(i)} \cdot b}, \quad \mathrm{wt}(\overline{b}) = w_0 \cdot \mathrm{wt}(b).$$

There is a crystal isomorphism $\overline{B_\lambda} \cong B_\lambda$. We compose this isomorphism with the map of sets $B_\lambda \to \overline{B_\lambda}$ given by $b \mapsto \overline{b}$ and denote the resulting map by $\xi = \xi_{B_\lambda} : B_\lambda \to B_\lambda$. We call the map ξ the *Schützenberger involution*. When $\mathfrak{g} = \mathfrak{gl}_n$, there is a realization of B_λ using tableaux. In this realization, ξ is the usual Schützenberger involution on tableaux (see [**14**]).

For an arbitrary $\mathfrak{g}$-crystal B, write $B = \bigoplus_{i=1}^k B_{\lambda_i}$. This is a decomposition of B into connected components. Then define $\xi_B : B \to B$ by $\xi_B = \bigoplus_{i=1}^k \xi_{B_{\lambda_i}}$. That is, we apply $\xi_{B_{\lambda_i}}$ to each connected component B_{λ_i}.

For crystals A and B, define

$$\sigma^S : A \otimes B \to B \otimes A, \quad \sigma^S(a \otimes b) = \xi_{B \otimes A}(\xi_B(b) \otimes \xi_A(a)).$$

THEOREM 6.1 ([**7**, Proposition 3, Theorem 3]). *We have*

(1) $\sigma^S_{B,A} \circ \sigma^S_{A,B} = \mathrm{id}$, *and*
(2) σ^S *satisfies the cactus relation* (4.1).

In other words, σ^S endows the category of $\mathfrak{g}$-crystals with the structure of a coboundary category.

6.3. The crystal commutor using the Kashiwara involution. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and let B_∞ be the $\mathfrak{g}$-crystal corresponding to the lower half $U_q^-(\mathfrak{g})$ of the associated quantized universal enveloping algebra. Let $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the $\mathbb{C}_q$-linear anti-automorphism given by

$$e_i^* = e_i, \quad f_i^* = f_i, \quad \left(q^h\right)^* = q^{-h}.$$

The map $*$ sends $U_q^-(\mathfrak{g})$ to $U_q^-(\mathfrak{g})$ and induces a map $* : B_\infty \to B_\infty$ (see [**12**, §8.3]). We call the map $*$ the *Kashiwara involution*.

Let B_λ be the $\mathfrak{g}$-crystal corresponding to the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ and let b_λ be its highest weight element. For two integral dominant weights λ and μ, there is an inclusion of crystals $B_{\lambda+\mu} \hookrightarrow B_\lambda \otimes B_\mu$ sending $b_{\lambda+\mu}$ to $b_\lambda \otimes b_\mu$. It follows from the tensor product rule that the image of this inclusion contains all elements of the form $b \otimes b_\mu$ for $b \in B_\lambda$. Thus we define a map

$$\iota_\lambda^{\lambda+\mu} : B_\lambda \to B_{\lambda+\mu}$$

which sends $b \in B_\lambda$ to the inverse image of $b \otimes b_\mu$ under the inclusion $B_{\lambda+\mu} \hookrightarrow B_\lambda \otimes B_\mu$. While this map is not a morphism of crystals, it is $\tilde{e}_i$-equivariant for all i and takes b_λ to $b_{\lambda+\mu}$.

The maps $\iota_\lambda^{\lambda+\mu}$ make the family of crystals B_λ into a directed system and the crystal B_∞ can be viewed as the limit of this system. We have $\tilde{e}_i$-equivariant maps $\iota_\lambda^\infty : B_\lambda \to B_\infty$ which we will simply denote by ι^∞ when it will cause no confusion. Define $\varepsilon^* : B_\infty \to P_+$ by

$$\varepsilon^*(b) = \min\{\lambda \mid b \in \iota^\infty(B_\lambda)\}$$

where we put the usual order on P_+, the positive weight lattice of $\mathfrak{g}$, given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. Recall that we also have the map $\varepsilon : B_\infty \to P_+$ given by $\varepsilon(b)(h_i) = \varepsilon_i(b)$. Then by [**12**, Proposition 8.2], the Kashiwara involution preserves weights and satisfies

(6.3) $$\varepsilon^*(b) = \varepsilon(b^*).$$

Consider the crystal $B_\lambda \otimes B_\mu$. Since $\varphi(b) = \varepsilon(b) + \text{wt}(b)$ for all $b \in B_\lambda$, we have that $\varphi(b_\lambda) = \text{wt}(b_\lambda) = \lambda$. It follows from the tensor product rule for crystals that the highest weight elements of $B_\lambda \otimes B_\mu$ are those elements of the form $b_\lambda \otimes b$ for $b \in B_\mu$ with $\varepsilon(b) \leq \lambda$. Thus $\varepsilon^*(b^*) = \varepsilon(b) \leq \lambda$ and so, by the definition of ε^*, we have $b^* \in \iota^\infty(B_\lambda)$. So we can consider b^* as an element of B_λ. Furthermore, $\varepsilon(b^*) = \varepsilon^*(b) \leq \mu = \varphi(b_\mu)$ since $b \in B_\mu$. Thus $b_\mu \otimes b^*$ is a highest weight element of $B_\mu \otimes B_\lambda$. Since $B_\lambda \otimes B_\mu \cong B_\mu \otimes B_\lambda$ as crystals, we can make the following definition.

DEFINITION 6.2 ([**9**, §3]). *Let* $\sigma_{B_\lambda, B_\mu}^c : B_\lambda \otimes B_\mu \xrightarrow{\cong} B_\mu \otimes B_\lambda$ *be the crystal isomorphism given uniquely by* $\sigma_{B_\lambda, B_\mu}^c(b_\lambda \otimes b) = b_\mu \otimes b^*$ *for* $b_\lambda \otimes b$ *a highest weight element of* $B_\lambda \otimes B_\mu$.

THEOREM 6.3 ([**9**, Theorem 3.1]). *For* $\mathfrak{g}$ *a simple complex Lie algebra,* $\sigma^S = \sigma^c$ *and so* σ^c *satisfies the cactus relation.*

We call σ^c the *crystal commutor*. Note that Theorem 6.3 only implies that it is a cactus commutor for $\mathfrak{g}$ of finite type.

6.4. The relationship between the various commutors. We have described three ways of constructing commutors in the categories of $U_q(\mathfrak{g})$-modules or $\mathfrak{g}$-crystals. The three definitions are closely related. In [**7**], Henriques and Kamnitzer defined a cactus commutor on the category of finite-dimensional $U_q(\mathfrak{g})$-modules when $\mathfrak{g}$ is of finite type, using an analogue of the Schützenberger involution on $U_q(\mathfrak{g})$. This definition of the commutor involves some choices of normalization. In [**10**], Kamnitzer and Tingley showed that Drinfel'd's commutor coming from the unitarized R-matrix corresponds to the commutor coming from the Schützenberger involution up to normalization. From there it follows that Drinfel'd's commutor preserves crystal lattices and acts on crystal bases as the crystal commutor, up to signs. The precise statement is the following.

PROPOSITION 6.4 ([**10**, Theorem 9.2]). *Suppose* (L_j, B_j) *are crystal bases of two finite-dimensional representations* V_j, $j = 1, 2$, *of* $U_q(\mathfrak{g})$ *for a simple complex Lie algebra* $\mathfrak{g}$. *Let* σ_{V_1, V_2}^D *be the isomorphism* $V_1 \otimes V_2 \cong V_2 \otimes V_1$ *given by* flip $\circ \bar{R}$. *Then*

$$\sigma_{V_1, V_2}^D(L_1 \otimes L_2) = L_2 \otimes L_1$$

and thus $\sigma_{L_1 \otimes L_2}^D$ *induces a map*

$$\sigma_{V_1 \otimes V_2}^{D \mod q^{-1/2}(L_1 \otimes L_2)} : (L_1 \otimes L_2)/q^{-1/2}(L_1 \otimes L_2) \to (L_2 \otimes L_1)/q^{-1/2}(L_2 \otimes L_1).$$

For all $b_j \in B_j$, $j = 1, 2$,

$$\sigma_{V_1 \otimes V_2}^{D \mod q^{-1/2}(L_1 \otimes L_2)}(b_1 \otimes b_2) = (-1)^{\langle \lambda + \mu - \nu, \rho^\vee \rangle} \sigma_{B_1, B_2}^c (b_1 \otimes b_2)$$

where λ, μ and ν are the highest weights of the connected components of B_1, B_2 and $B_1 \otimes B_2$ containing b_1, b_2 and $b_1 \otimes b_2$ respectively, ρ is half the sum of the positive roots of $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ denotes the pairing between the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and its dual $\mathfrak{h}^*$.

We thus have essentially two definitions of the crystal commutor. The first, using the Schützenberger involution (and coinciding with Drinfel'd's commutor in the crystal limit) only applies to $\mathfrak{g}$ of finite type but with this definition, it is apparent that the commutor satisfies the cactus relation. The second definition, using the Kashiwara involution, applies to $\mathfrak{g}$ of arbitrary type but it is not easy to see that it satisfies the cactus relation. In the next section, we will explain how a geometric interpretation of this commutor using quiver varieties allows one to prove that this is indeed the case.

7. A geometric realization of the crystal commutor

In this section we describe a geometric realization of the crystal commutor defined in Section 6.3 in the language of quiver varieties. This realization yields new insight into the coboundary structure and equips us with new geometric tools. Using these tools, one is able to show that the category of $\mathfrak{g}$-crystals for an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{g}$ can be given the structure of a coboundary category. This extends the previously known result, which held for $\mathfrak{g}$ of finite type.

7.1. Quiver varieties. Lusztig [16], Nakajima [20, 21, 22] and Malkin [19] have introduced varieties associated to quivers (directed graphs) built from the Dynkin graph of a Kac-Moody algebra $\mathfrak{g}$ with symmetric Cartan matrix. These varieties yield geometric realizations of the quantum group $U_q(\mathfrak{g})$, the representations of $\mathfrak{g}$, and tensor products of these representations in the homology (or category of perverse sheaves) of such varieties. In addition, Kashiwara and Saito [13, 23], Nakajima [22] and Malkin [19] have used quiver varieties to give a geometric realization of the crystals of these objects. Namely, they defined geometric operators on the sets of irreducible components of quiver varieties, endowing these sets with the structure of crystals. In the current paper, we will focus on the $\mathfrak{sl}_2$ case of these varieties for simplicity. In this case, the quiver varieties are closely related to grassmannians and flag varieties.

In this section, all vector spaces will be complex. Fix integers $w \geq 0$ and $n \geq 1$, and $\mathbf{w} = (w_i)_{i=1}^n \in (\mathbb{Z}_{\geq 0})^n$ such that $\sum_{i=1}^n w_i = w$. Let W be a w-dimensional vector space and let

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W, \quad \dim W_i / W_{i-1} = w_i \text{ for } 1 \leq i \leq n,$$

be an n-step partial flag in W. Define the *tensor product quiver variety*

$$\mathfrak{T}(\mathbf{w}) = \{(U, t) \mid U \subseteq W, \ t \in \operatorname{End} W, \ t(W_i) \subseteq W_{i-1} \ \forall i, \ \operatorname{im} t \subseteq U \subseteq \ker t\}.$$

We use the notation $\mathfrak{T}(\mathbf{w})$ since, up to isomorphism, this variety depends only on the dimensions of the subspaces W_i, $0 \leq i \leq n$. We have

$$\mathfrak{T}(\mathbf{w}) = \bigsqcup_{u=0}^{w} \mathfrak{T}(u, \mathbf{w}), \quad \text{where} \quad \mathfrak{T}(u, \mathbf{w}) = \{(U, t) \in \mathfrak{T}(\mathbf{w}) \mid \dim U = u\}.$$

Let $B(u, \mathbf{w})$ denote the set of irreducible components of $\mathfrak{T}(u, \mathbf{w})$ and set $B(\mathbf{w}) = \bigsqcup_u B(u, \mathbf{w})$.

Define

$$\mathrm{wt} : B(\mathbf{w}) \to P, \quad \mathrm{wt}(X) = w - 2u \text{ for } X \in B(u, \mathbf{w}),$$

$$\varepsilon : \mathfrak{T}(\mathbf{w}) \to \mathbb{Z}_{\geq 0}, \quad \varepsilon(U, t) = \dim U / \operatorname{im} t,$$

$$\varphi : \mathfrak{T}(\mathbf{w}) \to \mathbb{Z}_{\geq 0}, \quad \varphi(U, t) = \dim \ker t / U = \varepsilon(U, t) + w - 2 \dim U.$$

For $k \in \mathbb{Z}_{\geq 0}$, let

$$\mathfrak{T}(u, \mathbf{w})_k = \{(U, t) \in \mathfrak{T}(u, \mathbf{w}) \mid \varepsilon(U, t) = k\},$$

and for $X \in B(u, \mathbf{w})$, define $\varepsilon(X) = \varepsilon(U, t)$ and $\varphi(X) = \varepsilon(U, t)$ for a generic point (U, t) of X. Let

$$B(u, \mathbf{w})_k = \{X \in B(u, \mathbf{w}) \mid \varepsilon(X) = k\}, \quad B(\mathbf{w})_k = \bigsqcup_u B(u, \mathbf{w})_k.$$

The map

$$\mathfrak{T}(u, \mathbf{w})_k \to \mathfrak{T}(u - k, \mathbf{w})_0, \quad (U, t) \mapsto (\operatorname{im} t, t)$$

is a Grassmann bundle and thus induces an isomorphism

(7.1) $$B(u, \mathbf{w})_k \cong B(u - k, \mathbf{w})_0.$$

We then define crystal operators on $B(\mathbf{w})$ as follows. Suppose $X' \in B(u - k, \mathbf{w})_0$ corresponds to $X \in B(u, \mathbf{w})_k$ under the isomorphism (7.1). Define

$$\tilde{f}^k : B(u - k, \mathbf{w})_0 \to B(u, \mathbf{w})_k, \quad \tilde{f}^k(X') = X,$$

$$\tilde{e}^k : B(u, \mathbf{w})_k \to B(u - k, \mathbf{w})_0, \quad \tilde{e}^k(X) = X'.$$

For $k > 0$, we then define $\tilde{e}_i : B(\mathbf{w}) \to B(\mathbf{w})$ by

$$\tilde{e} : B(u, \mathbf{w})_k \xrightarrow{\tilde{e}^k} B(u - k, \mathbf{w})_0 \xrightarrow{\tilde{f}^{k-1}} B(u - 1; \mathbf{w})_{k-1},$$

and set $\tilde{e}_i(X) = 0$ for $X \in B(u, \mathbf{w})_0$. For $k > 2u - w$, define

$$\tilde{f} : B(u, \mathbf{w})_k \xrightarrow{\tilde{e}^k} B(u - k, \mathbf{w})_0 \xrightarrow{\tilde{f}^{k+1}} B(u + 1, \mathbf{w})_{k+1},$$

and set $\tilde{f}(X) = 0$ for $X \in B(u, \mathbf{w})_k$ with $k \leq 2u - w$. The maps $\tilde{e}^k$ and $\tilde{f}^k$ defined above can be considered as the kth powers of $\tilde{e}$ and $\tilde{f}$ respectively.

THEOREM 7.1 ([**22**, §7]). *The operators $\varepsilon, \varphi, \mathrm{wt}, \tilde{e},$ and $\tilde{f}$ endow the set $B(\mathbf{w})$ with the structure of an $\mathfrak{sl}_2$-crystal and $B(\mathbf{w}) \cong B_{w_1} \otimes \cdots \otimes B_{w_n}$ as $\mathfrak{sl}_2$-crystals.*

We let $\phi : B(\mathbf{w}) \cong B_{w_1} \otimes \cdots \otimes B_{w_n}$ denote the isomorphism of Theorem 7.1.

7.2. The geometric realization of the crystal commutor. Fix a hermitian form on W. Let $t^\dagger$ denote the hermitian adjoint of $t \in \operatorname{End} W$ and let $S^\perp$ denote the orthogonal complement to a subspace $S \subseteq W$. If we let $\hat{W}_i = W_{n-i}^\perp$ for $0 \leq i \leq n$, and $\hat{\mathbf{w}} = (\hat{w}_i)_{i=1}^n$ where

$$\hat{w}_i = \dim \hat{W}_i / \hat{W}_{i-1} = \dim W_{n-i}^\perp / W_{n-i+1}^\perp = w_{n-i+1},$$

then

$$\mathfrak{T}(\hat{\mathbf{w}}) = \{(U, t) \mid U \subseteq W, \ t \in \operatorname{End} W, \ t(\hat{W}_i) \subseteq \hat{W}_{i-1} \ \forall i, \ \operatorname{im} t \subseteq U \subseteq \ker t\}.$$

Note that $\varepsilon(U, t) = 0$ if and only if $U = \operatorname{im} t$. Also, for $t \in \operatorname{End} W$,

$$t(W_i) \subseteq W_{i-1} \Rightarrow t^\dagger(\hat{W}_{n-i+1}) \subseteq \hat{W}_{n-i}.$$

Therefore,
$$(\operatorname{im} t, t) \in \mathfrak{T}(\mathbf{w}) \iff (\operatorname{im} t^\dagger, t^\dagger) \in \mathfrak{T}(\hat{\mathbf{w}}),$$
and the map $(\operatorname{im} t, t) \mapsto (\operatorname{im} t^\dagger, t^\dagger)$ induces isomorphisms
$$\mathfrak{T}(u, \mathbf{w})_0 \cong \mathfrak{T}(u, \hat{\mathbf{w}})_0, \quad B(u, \mathbf{w})_0 \cong B(u, \hat{\mathbf{w}})_0.$$
We denote the isomorphism $B(u, \mathbf{w})_0 \cong B(u, \hat{\mathbf{w}})_0$ by $X \mapsto X^\dagger$ for $X \in B(u, \mathbf{w})_0$. Since the elements of $B(\mathbf{w})_0$ are precisely the highest weight elements of the crystal $B(\mathbf{w})$, a commutor is uniquely determined by its action on these elements.

THEOREM 7.2 ([**24**, §4.2]).   (1) *If $n = 2$ and $X \in B(\mathbf{w})_0$, we have*
$$\phi^{-1} \circ \sigma^c_{B_{w_1}, B_{w_2}} \circ \phi(X) = X^\dagger,$$

and thus the map $X \mapsto X^\dagger$ corresponds to the crystal commutor on highest weight elements.
   (2) *If $n = 3$ and $X \in B(\mathbf{w})_0$,*

$$\phi^{-1} \circ \left(\sigma^c_{B_{w_1}, B_{w_3} \otimes B_{w_2}} \circ \left(\operatorname{id}_{B_{w_1}} \otimes \sigma^c_{B_{w_2} \otimes B_{w_3}} \right) \right) \circ \phi(X) = X^\dagger$$
$$= \phi^{-1} \circ \left(\sigma^c_{B_{w_2} \otimes B_{w_1}, B_{w_3}} \circ \left(\sigma^c_{B_{w_1}, B_{w_2}} \otimes \operatorname{id}_{B_{w_3}} \right) \right) \circ \phi(X),$$

and thus the crystal commutor satisfies the cactus relation.

One advantage of the geometric interpretation of the crystal commutor defined here is that it extends to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. In particular, if $\mathfrak{g}$ has symmetric Cartan matrix, then there exists a tensor product quiver variety whose irreducible components can be given the structure of a tensor product crystal. There then exists a map $X \mapsto X^\dagger$, which generalizes the map defined above. One can show that, in the case of two factors, this map corresponds to the crystal commutor. For three factors, the compositions $\sigma^c_{B_{\lambda_1}, B_{\lambda_3} \otimes B_{\lambda_2}} \circ \left(\operatorname{id}_{B_{\lambda_1}} \otimes \sigma^c_{B_{\lambda_2} \otimes B_{\lambda_3}} \right)$ and $\sigma^c_{B_{\lambda_2} \otimes B_{\lambda_1}, B_{\lambda_3}} \circ \left(\sigma^c_{B_{\lambda_1}, B_{\lambda_2}} \otimes \operatorname{id}_{B_{\lambda_3}} \right)$ both correspond (on highest weight elements) to the map $X \mapsto X^\dagger$ and are therefore equal. Thus the commutor satisfies the cactus relation. When $\mathfrak{g}$ is symmetrizable but with non-symmetric Cartan matrix, one can use a well-known folding argument to obtain the same result from the symmetric case. We therefore have the following theorem.

THEOREM 7.3 ([**24**, Theorem 6.4]). *For a symmetrizable Kac-Moody algebra $\mathfrak{g}$, the category of $\mathfrak{g}$-crystals is a coboundary monoidal category with cactus commutor σ^c.*

This generalizes the previously known result for $\mathfrak{g}$ of finite type. We refer the reader to [**24**] for details.

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