Graphical Calculus for the Hecke Algebra

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Abstract

We develop a graphical calculus for the Hecke algebra of type A, when its parameter is nonzero and not a nontrivial root of unity. Then we proceed to generalize the concepts of symmetrizer and antisymmetrizer to this algebra. We also explore some of their properties, recursive relations and their interaction with each other.

1 Introduction

The symmetric group $\mathfrak{S}_n$ is the group of permutations of the set $\{1, 2, \ldots, n\}$. The study of permutations and combinatorics dates back to the 6th century B.C. in India, where the first traces of this kind of thinking can be found. Their importance grew substantially in 1770 when Lagrange published "Réflexions sur la Résolution Algébrique des Équations". This paper initiated the study of permutations in connection with the study of the solution of polynomial equations. It was probably the first clear instance of implicit group-theoretic thinking in mathematics. It led directly to the works of P. Ruffini, N. Abel and E. Galois during the first third of the 19th century, and the concept of permutation group and, in particular, the symmetric group. Later in that century A. Cayley defined the modern concept of a group and proved that every finite group is isomorphic to a subgroup of the symmetric group, a result known as Cayley's Theorem.

The finite dimensional irreducible representations of $\mathfrak{S}_n$ in characteristic zero are very well understood, with everything from their degrees and character formulae to explicit matrix representation being known for many years. In contrast, very little detailed information is known about the modular representation theory of the symmetric groups.

In this work we are concerned with the Hecke algebra (or Iwahori-Hecke algebra) of type A which is a q-deformation of the group algebra of the symmetric group. Loosely speaking, it is the quantum analogue of the symmetric group that corresponds to passing from Lie algebras to quantum groups. When q is nonzero and not a nontrivial root of unity, then its representation theory is very similar to the representation theory of the symmetric group. At a nontrivial root of unity, its representation theory is more similar to the modular representation theory of the symmetric group in characteristic zero.

The study of Hecke algebras was pioneered by R. Dipper and G. James in a series of landmark papers (see [5] for references and a more detailed overview of the history of this subject). The main motivation for the study of Hecke algebras is that they provide a bridge between the representation theory of the symmetric and general linear groups. In addition, Hecke algebras play an important role in the theory of categorification. They also have applications to many other areas of algebra, combinatorics and geometric representation theory.

In the book [2] we find a graphical calculus using string diagrams for symmetrizers and antisymmetrizers defined on the symmetric group. The goal of this work is to generalize these notions for the Hecke algebra together with their graphical description. The graphical calculus is a powerful tool for theoretical and practical purposes. From a practical point of view, since we are very visual beings, it helps us to visualize relations and properties more easily, which may not even be noticed without it, giving a hand to intuition. For instance, in the graphical calculus, one of the defining relations of the Hecke algebra (relation (b) in Definition 1) follows immediately from the fact that we consider diagrams up to isotopy relative the boundary. Another feature of the graphical calculus is that, since the group algebra of the symmetric group and the Hecke algebra are quotients of the group algebra of the braid group, their graphical calculi share deep similarities. From the theoretical point of view there is interest in diagrammatic categorification in which Hecke algebras play an important role. For instance, the graphical calculus of the Hecke algebra is of fundamental importance in the categorification of the Heisenberg algebra developed in [4]. In fact, in the current paper we will provide proofs of some of the statements asserted there without proof.

Acknowledgments. I would like to thank the Mitacs Globalink Program, the University of Ottawa and the Capes for providing such an amazing oportunity for undergraduate students around the world. I would also like to thank A. Savage for guiding me through the subject and the staff of the mathematics department of UNESP - Rio Claro for the help and incentive. Finally, I would like to thank N. Manning for helpful discussions and suggestions.

2 Graphical Description

In our work we are going to explore properties of the symmetrizer, antisymmetrizer and their relation. All these entities are defined on the Hecke algebra.

Definition 1 (Hecke algebra.)**.** *Let $\mathbb{F}$ be a field of characteristic zero and consider $q \neq 0$ an element of $\mathbb{F}$, which is not a nontrivial root of unity. For $n \geq 2$, the Hecke algebra $H_n(q)$ is the $\mathbb{F}$-algebra generated by $t_1, \dots, t_{n-1}$ with relations*

(a) $t_i^2 = q + (q-1)t_i$ *for* $i = 1, 2, \dots, n-1$,

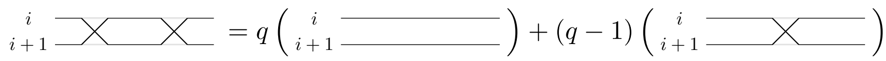
(b) $t_i t_j = t_j t_i$ *for* $i, j = 1, 2, \dots, n-1$ *such that* $|j - i| > 1$,

(c) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ *for* $i = 1, 2, \dots, n-2$.

By convention, we set $H_0(q) = H_1(q) = \mathbb{F}$. We let 1 denote the identity element of $H_n(q)$. The algebra H_n has a basis $\{t_\sigma\}_{\sigma \in \mathfrak{S}_n}$ where for $\sigma \in \mathfrak{S}_n$, $t_\sigma = t_{i_1} \cdots t_{i_k}$, where $\sigma = s_{i_1} \cdots s_{i_k}$ is a reduced expression for σ.

Throughout our work we shall make use of a graphical description of the Hecke algebra formalized below. The graphical description of the Hecke algebra is the $\mathbb{F}$-module generated by planar diagrams modulo local relations (Hecke algebra relations). The diagrams are oriented compact 1-manifolds immersed in the strip $[0,1] \times \mathbb{R}$, modulo rel boundary isotopies. The endpoints of the 1-manifold are located at $\{0\} \times \{1, 2, \dots, n\}$ and $\{1\} \times \{1, 2, \dots, n\}$. In such a description, the element t_i is represent by crossing the ith and $(i+1)$th lines. There is also a natural product defined by gluing diagrams. The local relations are as follows.

(1)



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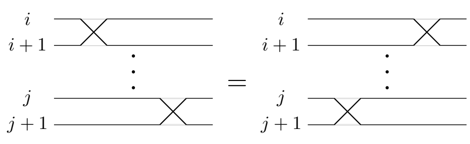

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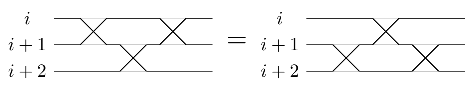
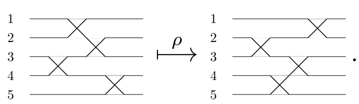


Notice that if we set $q = 1$ in Definition 1, then $H_n(1)$ is the group algebra of the symmetric group $\mathfrak{S}_n$.

Definition 2. *We define the homomorphism $\rho : H_n(q) \to H_n(q)$ by $\rho(t_i) = t_{n-i}$. This homomorphism is in fact a automorphism called the 'flipping' automorphism. Diagrammatically, this automorphism performs a flip in the horizontal axis. For example, the image of the element $t_2 t_4 t_3 t_1$ under the flipping automorphism is $t_3 t_1 t_2 t_4$.*

(4)



3 The Symmetrizer and The Antisymmetrizer

Definition 3. *In $H_n(q)$, let m, k be positive integers such that $k \leq k+m-1 \leq n$. Define the homomorphism $\iota_k^m : \mathfrak{S}_m \to \mathfrak{S}_n$ by*

$$\iota_k^m(\sigma)(t) = \begin{cases} (k-1) + \sigma(t + (1-k)), & t \in \{k, k+1, \dots, k+(m-1)\}, \\ t, & t \in \{1, 2, \dots, k-1, k+m, \dots, n\}. \end{cases} \tag{5}$$

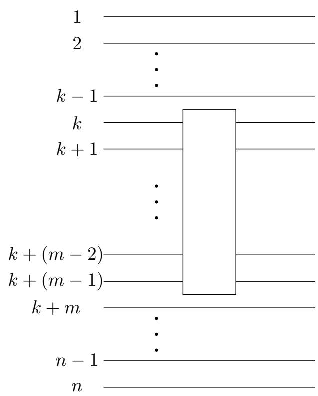
We define the symmetrizer to be

$$S_{k,k+1,\dots,k+(m-1)} = \frac{1}{[m]_q!} \sum_{\sigma \in \mathfrak{S}_m} t_{\iota_k^m(\sigma)}. \tag{6}$$

Where $[n]_q! = [1]_q[2]_q \cdots [n-1]_q[n]_q$ and

$$[n]_q = \sum_{i=0}^{n-1} q^i. \tag{7}$$

It is depicted by a white box enveloping the strings from k to $k+(m-1)$.

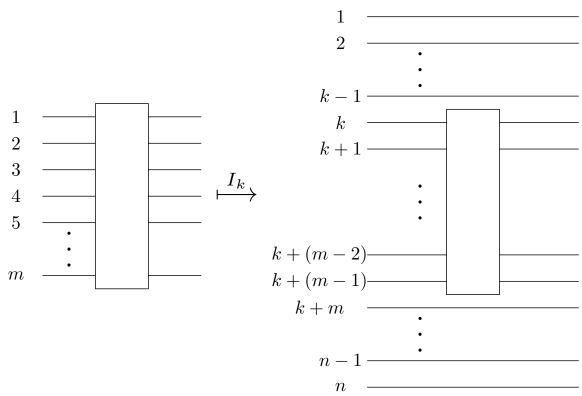


When $k = 1$ and $m = n$ we shall denote the symmetrizer $S_{1,2,\dots,n}$ by S.

Remark 1. Given k, m, n positive integers such that $k + m - 1 \leq n$, there exists a natural embedding of $H_m(q)$ into $H_n(q)$ with relation to the kth string given by

$$\begin{aligned} I_k : H_m(q) &\to H_n(q) \\ t_i &\mapsto t_{k+i-1}. \end{aligned}$$

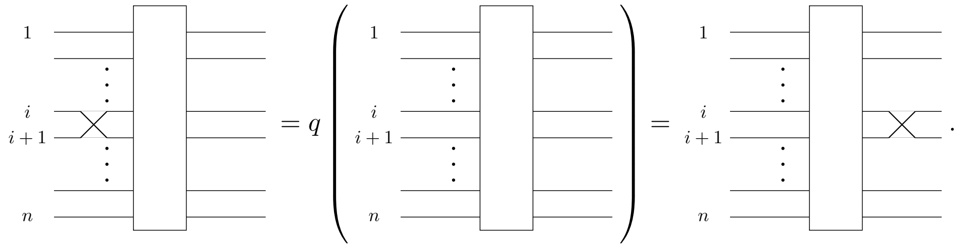
In view of this the following statements for the symmetrizer S and for the (to be defined) antisymmetrizer A are also valid for $S_{k,\dots,k+m-1}$ and $A_{k,\dots,k+m-1}$.



The next result tells us that crossings may be absorbed by the symmetrizer at the cost of a factor q.

Theorem 1. *For all $i \in \{1, 2, \ldots, n-1\}$ we have $t_i S = qS = St_i$.*

(8)



Proof. Let $i \in \{1, 2, \ldots, n-1\}$ be a fixed integer. Let $A = \{\sigma \in \mathfrak{S}_n; \ell(s_i\sigma) \geq \ell(\sigma)\}$ and $B = \{\sigma \in \mathfrak{S}_n; \ell(s_i\sigma) < \ell(\sigma)\}$. Let $f : A \to B$ be the function defined by $f(\sigma) = s_i\sigma$. We must prove that f is a bijection. In fact, clearly f is injective, to show that f is onto let $\tau \in B$. The permutation $s_i\tau$ is in A since $\ell(s_i(s_i\tau)) = \ell(\tau) > \ell(s_i\tau)$, furthermore $f(s_i\tau) = \tau$. In view of this we may rewrite the symmetrizer as

$$S = \frac{1}{[n]_q!}\sum_{\sigma\in A}(t_\sigma + t_{f(\sigma)}). \tag{9}$$

On the other hand,

$$t_i(t_\sigma + t_{f(\sigma)}) = t_i t_\sigma + t_i t_{f(\sigma)} = t_{f(\sigma)} + {t_i}^2 t_\sigma = t_{f(\sigma)} + qt_\sigma + (q-1)t_i t_\sigma = q(t_{f(\sigma)} + t_\sigma), \tag{10}$$

where we have used that $t_i t_\sigma = t_{s_i\sigma}$, since $\ell(s_i\sigma) = \ell(\sigma) + \ell(s_i)$ for $\sigma \in A$. Therefore we have

$$t_i S = \frac{1}{[n]_q!}\sum_{\sigma\in A} t_i(t_\sigma + t_{f(\sigma)}) = \frac{1}{[n]_q!}\sum_{\sigma\in A} q(t_{f(\sigma)} + t_\sigma) = qS. \tag{11}$$

The other equality follows by symmetry. $\square$

Lemma 1. **(a)** *For a given $i \in \{1, 2, \ldots, n+1\}$, let $\mathfrak{S}_{n+1}^{(i)} = \{\tau \in \mathfrak{S}_{n+1}; \tau(n+1) = i\}$. Then*

$$\sum_{\tau\in\mathfrak{S}_{n+1}^{(i)}} q^{\ell(\tau)} = q^{n+1-i}\sum_{\tau\in\mathfrak{S}_n} q^{\ell(\tau)}. \tag{12}$$

(b) *For a given positive integer n we have the following identity,*

$$[n]_q! = \sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)}. \tag{13}$$

Proof. **(a)** In fact, it is easy to see that equation (12) holds for $i = n+1$. Since the correspondence $\phi \mapsto \psi\phi$, where $\psi = (i, i+1, \ldots, n, n+1)$, between $\mathfrak{S}_{n+1}^{(n+1)}$ and $\mathfrak{S}_{n+1}^{(i)}$ is a bijection we can write

$$\sum_{\tau\in\mathfrak{S}_{n+1}^{(i)}} q^{\ell(\tau)} = \sum_{\phi\in\mathfrak{S}_{n+1}^{(n+1)}} q^{\ell(\psi\phi)}. \tag{14}$$

We claim that $\ell(\psi\phi) = \ell(\psi) + \ell(\phi)$. Consider the sets Φ of the inversions of the permutation ϕ and $\Psi = \{(j, n+1); n+1 > \phi(j) \geq i\}$. These sets are disjoint since $\phi(n+1) = n+1$ and it is a straightforward case-by-case analysis to check that both are subsets of the set of inversions of $\psi\phi$, whence we have that

$$\ell(\psi\phi) \geq \#(\Psi \cup \Phi) = \ell(\psi) + \ell(\phi).$$

Since $\ell(\psi\phi) \leq \ell(\psi) + \ell(\phi)$ we conclude that $\ell(\psi\phi) = \ell(\psi) + \ell(\phi)$. Then equation (14) becomes

$$\sum_{\tau\in\mathfrak{S}_{n+1}^{(i)}} q^{\ell(\tau)} = \sum_{\phi\in\mathfrak{S}_{n+1}^{(n+1)}} q^{\ell(\psi)+\ell(\phi)} = q^{\ell(\psi)}\sum_{\phi\in\mathfrak{S}_{n+1}^{(n+1)}} q^{\ell(\phi)} = q^{n+1-i}\sum_{\phi\in\mathfrak{S}_n} q^{\ell(\phi)}. \tag{15}$$

(b) We shall prove by induction on n. For $n=1$ it is trivial. Suppose the equality holds for some positive integer n and we shall prove it holds for $n+1$. Let $\mathfrak{S}_{n+1}^{(i)} = \{\tau \in \mathfrak{S}_{n+1}; \tau(n+1) = i\}$. It is easy to see that these sets are disjoint and their union is equal to $\mathfrak{S}_{n+1}$. Thus we may write:

$$\sum_{\tau\in\mathfrak{S}_{n+1}} q^{\ell(\tau)} = \sum_{\tau\in\mathfrak{S}_{n+1}^{(1)}} q^{\ell(\tau)} + \sum_{\tau\in\mathfrak{S}_{n+1}^{(2)}} q^{\ell(\tau)} + \cdots + \sum_{\tau\in\mathfrak{S}_{n+1}^{(n)}} q^{\ell(\tau)} + \sum_{\tau\in\mathfrak{S}_{n+1}^{(n+1)}} q^{\ell(\tau)}. \tag{16}$$

By Lemma 1(a) we have that

$$\sum_{\tau\in\mathfrak{S}_{n+1}^{(i)}} q^{\ell(\tau)} = q^{n+1-i} \sum_{\tau\in\mathfrak{S}_{n}} q^{\ell(\tau)} \tag{17}$$

then, using our induction hypothesis that

$$[n]_q! = \sum_{\tau\in\mathfrak{S}_{n}} q^{\ell(\tau)},$$

we can rewrite equation (16) to obtain

$$\sum_{\tau\in\mathfrak{S}_{n+1}} q^{\ell(\tau)} = q^n[n]_q! + q^{n-1}[n]_q! + \cdots + q[n]_q! + [n]_q! = (q^n + q^{n-1} + \cdots + 1)[n]_q! = [n+1]_q[n]_q! = [n+1]_q!. \tag{18}$$

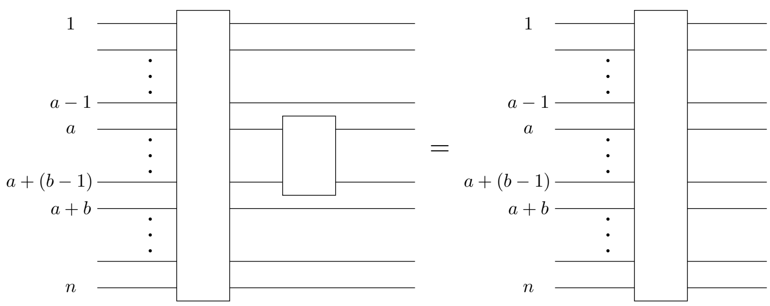
□

The next result tells us that bigger symmetrizers absorb smaller ones.

Theorem 2. *If $1 \le a < a+b \le n+1$, then*

$$SS_{a,\dots,a+(b-1)} = S_{k,\dots,k+(m-1)} = S_{a,\dots,a+(b-1)}S. \tag{19}$$

(20)



Proof. By definition of $S_{a,\dots,a+(b-1)}$ we have,

$$S_{a,\dots,a+(b-1)}S = \left(\frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} t_{\iota_a^b(\sigma)}\right) S = \frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} (t_{\iota_a^b(\sigma)}S). \tag{21}$$

Now, applying Theorem 1 we obtain,

$$\frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} (t_{\iota_a^b(\sigma)}S) = \frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} (q^{\ell(\sigma)}S). \tag{22}$$

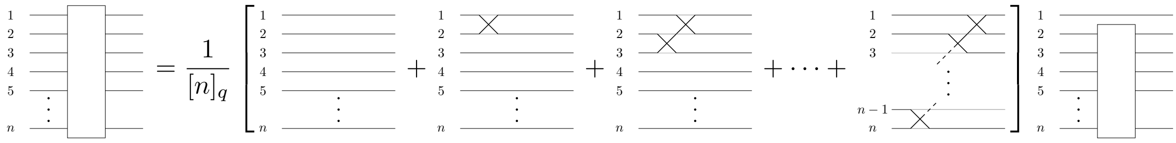
Finally, by the distributive law and Lemma 1(b),

$$\frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} (q^{\ell(\sigma)}S) = \left(\frac{1}{[b]_q!}\sum_{\sigma\in\mathfrak{S}_b} q^{\ell(\sigma)}\right) S = S. \tag{23}$$

□

Theorem 3. *We have the following recursive relation*

$$S = \frac{1}{[n]_q}[1 + t_1 + t_2t_1 + t_3t_2t_1 + \cdots + t_{n-1}t_{n-2}\cdots t_1]S_{2,\ldots,n}. \tag{24}$$

(25) 

Proof. By definition of $S_{2,\ldots,n}$ we have that

$$\frac{1}{[n]_q}[1 + t_1 + t_2t_1 + \cdots + t_{n-1}\cdots t_1]S_{2,\ldots,n} = \frac{1}{[n]_q!}[1 + t_1 + t_2t_1 + \cdots + t_{n-1}\cdots t_1]\sum_{\sigma\in\mathfrak{S}_{n-1}} t_{\iota_2^{n-1}(\sigma)}. \tag{26}$$

As we have seen in the proof of Lemma 1(a), for $i = 0, 1, \ldots, n-2$ and $\sigma \in \mathfrak{S}_{n-1}$,

$$t_{1+i}t_i\cdots t_1 t_{\iota_2^{n-1}(\sigma)} = t_{s_{1+i}s_i\cdots s_1\iota_2^{n-1}(\sigma)}. \tag{27}$$

Furthermore, the sets $\iota_2^{n-1}(\mathfrak{S}_{n-1}), s_1\iota_2^{n-1}(\mathfrak{S}_{n-1}), s_2s_1\iota_2^{n-1}(\mathfrak{S}_{n-1}), \ldots, s_{n-1}s_{n-2}\cdots s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})$ are pairwise disjoint, since they each fix different subsets of $\{1, 2, \ldots, n\}$, and their union equals $\mathfrak{S}_n$ since we have $n!$ distinct elements of $\mathfrak{S}_n$.

Then we can write equation (26) as

$$\begin{aligned}
&\frac{1}{[n]_q!}[1 + t_1 + t_2t_1 + \cdots + t_{n-1}\cdots t_1]\sum_{\sigma\in\mathfrak{S}_{n-1}} t_{\iota_2^{n-1}(\sigma)}\\
&= \frac{1}{[n]_q!}\left[\sum_{\sigma\in\iota_2^{n-1}(\mathfrak{S}_{n-1})} t_\sigma + \sum_{\sigma\in s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})} t_\sigma + \cdots + \sum_{\sigma\in s_{n-1}\cdots s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})} t_\sigma\right]\\
&= \frac{1}{[n]_q!}\sum_{\sigma\in\mathfrak{S}_n} t_\sigma\\
&= S. \qquad (28)
\end{aligned}$$

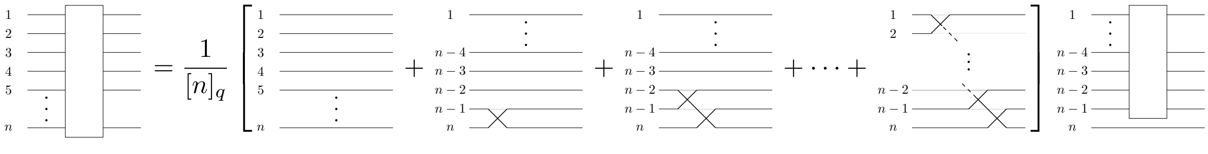
By (26) and (28) we conclude that

$$S = \frac{1}{[n]_q}[1 + t_1 + t_2t_1 + t_3t_2t_1 + \cdots + t_{n-1}t_{n-2}\cdots t_1]S_{2,\ldots,n}. \tag{29}$$

□

Corollary 1. *We have the following recursive relation*

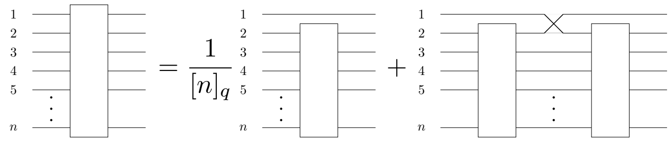
$$S = \frac{1}{[n]_q}[1 + t_{n-1} + t_{n-2}t_{n-1} + t_{n-3}t_{n-2}t_{n-1} + \cdots + t_1t_2\cdots t_{n-1}]S_{1,\ldots,n-1}. \tag{30}$$

(31) 

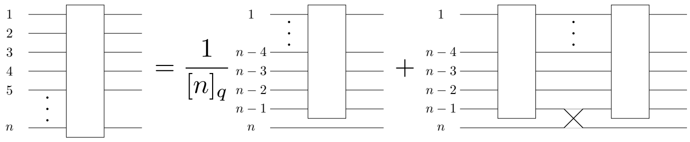
Proof. The image of S under the flipping automorphism ρ is S. Thus applying ρ on both sides of equation (24) of Theorem 3 we obtain equation (30). □

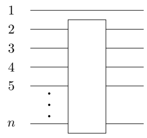
Corollary 2. *We have the following relations:*

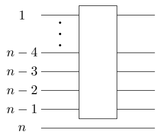
(i)

(32) 

(ii)

(33) 

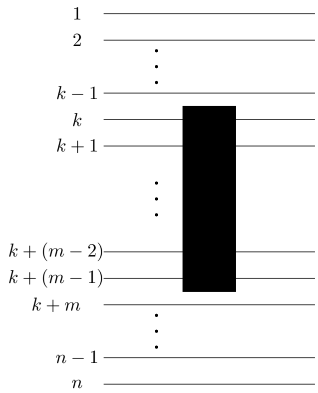
Proof. **(i)** Multiply equation (25), on the left, by  and use Theorem 1.

(ii) Multiply equation (31), on the left, by  and use Theorem 1. □

Definition 4. *Let m, k be positive integers such that $k \leq k+m-1 \leq n$. We define the antisymmetrizer to be*

(34)
$$A_{k,k+1,\ldots,k+(m-1)} = \frac{1}{[m]_{q^{-1}}!} \sum_{\sigma \in \mathfrak{S}_m} (-q)^{-\ell(\sigma)} t_{\iota_k^m(\sigma)}.$$

It is depicted by a black box enveloping the strings from k to $k+(m-1)$.

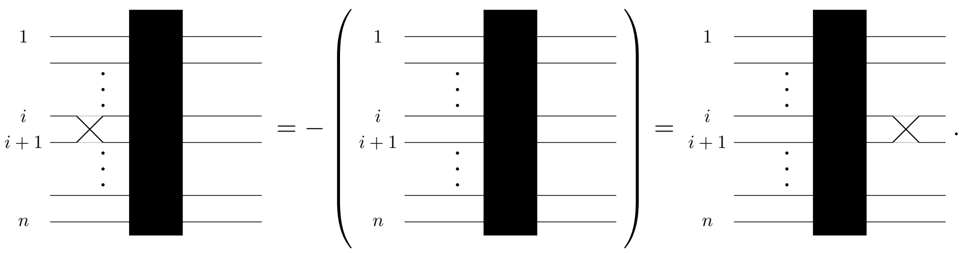


When $k = 1$ and $m = n$ we shall denote the antisymmetrizer $A_{1,2,\ldots,n}$ by A.

As noted in Remark 1 all the statements below about the antisymmetrizer A are also valid for $A_{k,\ldots,k+m-1}$.

The next result tells us that crossings may be absorbed by the antisymmetrizer at the cost of a sign.

Theorem 4. *For all $i \in \{1, 2, \ldots, n-1\}$ we have $t_i A = -A = A t_i$.*

(35) 

Proof. Let $i \in \{1, 2, \ldots, n-1\}$ be a fixed integer. Let $L = \{\sigma \in \mathfrak{S}_n; \ell(s_i\sigma) \geq \ell(\sigma)\}$ and $B = \{\sigma \in \mathfrak{S}_n; \ell(s_i\sigma) < \ell(\sigma)\}$. Let $f : L \to B$ be the function defined by $f(\sigma) = s_i\sigma$. As we have seen in the proof of Theorem 1, f is a bijection. In view of this we may rewrite the antisymmetrizer as

$$A = \frac{1}{[n]_{q^{-1}}!}\sum_{\sigma\in A}((-q)^{-\ell(\sigma)}t_\sigma + (-q)^{-\ell(f(\sigma))}t_{f(\sigma)}). \tag{36}$$

On the other hand,

$$\begin{aligned} t_i((-q)^{-\ell(\sigma)}t_\sigma + (-q)^{-\ell(f(\sigma))}t_{f(\sigma)}) = (-q)^{-\ell(\sigma)}t_it_\sigma + (-q)^{-\ell(f(\sigma))}t_it_{f(\sigma)} \\ =(-q)^{-\ell(\sigma)}t_{f(\sigma)} + (-q)^{-\ell(f(\sigma))}{t_i}^2t_\sigma \\ =- q(-q)^{-\ell(f(\sigma))}t_{f(\sigma)} + (-q)^{-\ell(f(\sigma))}(qt_\sigma + (q-1)t_it_\sigma) \\ =- q(-q)^{-\ell(f(\sigma))}t_{f(\sigma)} - (-q)^{-\ell(\sigma)}t_\sigma - (-q)^{-\ell(f(\sigma))}(1-q)t_it_\sigma) \\ =- (t_{f(\sigma)} + t_\sigma), \end{aligned} \tag{37}$$

where we have used that $t_it_\sigma = t_{s_i\sigma}$, since $\ell(s_i\sigma) = \ell(\sigma) + \ell(s_i)$ for $\sigma \in A$. Therefore we have

$$\begin{aligned} t_iA = \frac{1}{[n]_{q^{-1}}!}\sum_{\sigma\in A}t_i((-q)^{-\ell(\sigma)}t_\sigma + (-q)^{-\ell(f(\sigma))}t_{f(\sigma)}) \\ = \frac{1}{[n]_{q^{-1}}!}\sum_{\sigma\in A}-((-q)^{-\ell(\sigma)}t_\sigma + (-q)^{-\ell(f(\sigma))}t_{f(\sigma)}) = -A. \end{aligned} \tag{38}$$

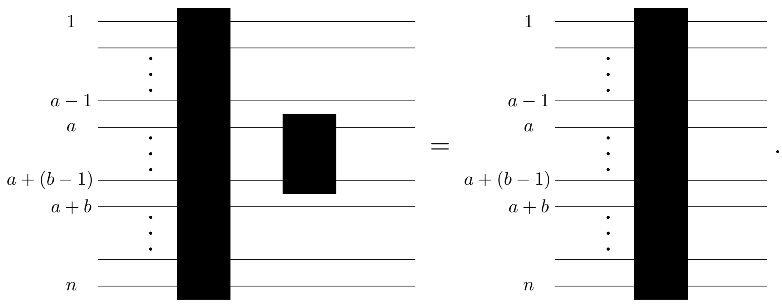
The other equality follows by symmetry. $\square$

The next result tells us that bigger antisymmetrizers absorb smaller ones.

Theorem 5. *If $1 \leq a < b + a \leq n + 1$, then*

$$AA_{a,\ldots,a+(b-1)} = A = A_{a,\ldots,a+(b-1)}A. \tag{39}$$

(40)



Proof. By definition of $A_{a,\ldots,a+(b-1)}$ we have,

$$A_{a,\ldots,a+(b-1)}A = \left(\frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}(-q)^{-\ell(\sigma)}t_{\iota_a^b(\sigma)}\right)A = \frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}((-q)^{-\ell(\sigma)}t_{\iota_a^b(\sigma)}A). \tag{41}$$

Now, applying Theorem 4 we obtain,

$$\frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}((-q)^{-\ell(\sigma)}t_{\iota_a^b(\sigma)}A) = \frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}((-q)^{-\ell(\sigma)}(-1)^{\ell(\sigma)}A) = \frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}(q^{-\ell(\sigma)}A). \tag{42}$$

Finally, by the distributive law, together with the fact that

$$[b]_{q^{-1}}! = \sum_{\sigma\in\mathfrak{S}_b}q^{-\ell(\sigma)}, \tag{43}$$

which is Lemma 1(b) after replacing q by q^{-1}, we have

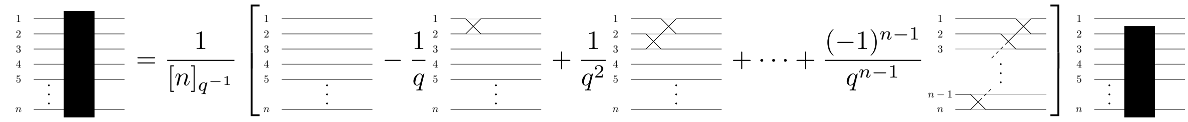
$$\frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}(q^{-\ell(\sigma)}A) = \left(\frac{1}{[b]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_b}q^{-\ell(\sigma)}\right)A = A. \tag{44}$$

$\square$

Theorem 6. *We have the following recursive relation*

$$A = \frac{1}{[n]_{q^{-1}}}[1 - q^{-1}t_1 + q^{-2}t_2t_1 - q^{-3}t_3t_2t_1 + \cdots + (-q)^{1-n}t_{n-1}t_{n-2}\cdots t_1]A_{2,\dots,n}. \tag{45}$$

(46)



Proof. By definition of $A_{2,\dots,n}$ we have that

$$\frac{1}{[n]_{q^{-1}}}[1 + (-q)^{-1}t_1 + (-q)^{-2}t_2t_1 + \cdots + (-q)^{1-n}t_{n-1}\cdots t_1]A_{2,\dots,n} = \tag{47}$$
$$\frac{1}{[n]_{q^{-1}}!}[1 + (-q)^{-1}t_1 + (-q)^{-2}t_2t_1 + \cdots + (-q)^{1-n}t_{n-1}\cdots t_1] \sum_{\sigma\in\mathfrak{S}_{n-1}} (-q)^{-\ell(\sigma)} t_{\iota_2^{n-1}(\sigma)}.$$

As we have seen in the proof of Lemma 1(a), for $i = 0, 1, \dots, n-2$ and $\sigma \in \mathfrak{S}_{n-1}$,

$$t_{1+i}t_i\cdots t_1 t_{\iota_2^{n-1}(\sigma)} = t_{s_{1+i}s_i\cdots s_1\iota_2^{n-1}(\sigma)}. \tag{48}$$

Furthermore, the sets $\iota_2^{n-1}(\mathfrak{S}_{n-1}), s_1\iota_2^{n-1}(\mathfrak{S}_{n-1}), s_2s_1\iota_2^{n-1}(\mathfrak{S}_{n-1}), \dots, s_{n-1}s_{n-2}\cdots s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})$ are pairwise disjoint, since they each fix different subsets of $\{1, 2, \dots, n\}$, and their union equals $\mathfrak{S}_n$ since we have $n!$ distinct elements of $\mathfrak{S}_n$.

Then we can write equation (47) as

$$\begin{aligned}
&\frac{1}{[n]_{q^{-1}}!}[1 + (-q)^{-1}t_1 + (-q)^{-2}t_2t_1 + \cdots + (-q)^{1-n}t_{n-1}\cdots t_1] \sum_{\sigma\in\mathfrak{S}_{n-1}} (-q)^{-\ell(\sigma)} t_{\iota_2^{n-1}(\sigma)} \\
&= \frac{1}{[n]_{q^{-1}}!}\left[\sum_{\sigma\in\iota_2^{n-1}(\mathfrak{S}_{n-1})} (-q)^{-\ell(\sigma)}t_\sigma + \sum_{\sigma\in s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})} (-q)^{-\ell(\sigma)}t_\sigma + \cdots + \sum_{\sigma\in s_{n-1}\cdots s_1\iota_2^{n-1}(\mathfrak{S}_{n-1})} (-q)^{-\ell(\sigma)}t_\sigma\right] \\
&= \frac{1}{[n]_{q^{-1}}!}\sum_{\sigma\in\mathfrak{S}_n} (-q)^{-\ell(\sigma)}t_\sigma \\
&= A. \qquad (49)
\end{aligned}$$

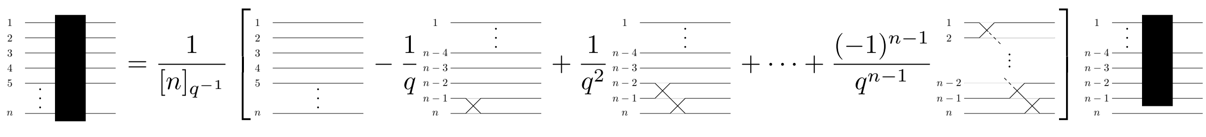
By (47) and (49) we conclude that

$$A = \frac{1}{[n]_{q^{-1}}}[1 - q^{-1}t_1 + q^{-2}t_2t_1 - q^{-3}t_3t_2t_1 + \cdots + (-q)^{1-n}t_{n-1}t_{n-2}\cdots t_1]A_{2,\dots,n}. \tag{50}$$ □

Corollary 3. *We have the following recursive relation*

$$A = \frac{1}{[n]_{q^{-1}}}[1 - q^{-1}t_{n-1} + q^{-2}t_{n-2}t_{n-1} - q^{-3}t_{n-3}t_{n-2}t_{n-1} + \cdots + (-q)^{1-n}t_1t_2\cdots t_{n-1}]A_{1,\dots,n-1}. \tag{51}$$

(52)

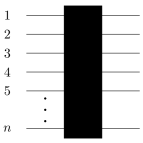
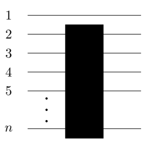
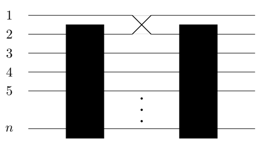


Proof. It is easy to check that the image of A under the flipping automorphism ρ is A. Thus applying ρ on both sides of equation (45) of Theorem 6 we obtain equation (51). □

Corollary 4. *We have the following relations:*

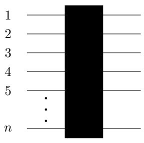
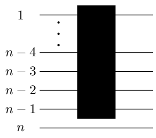
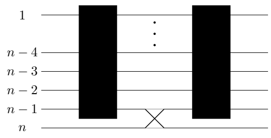
(i)

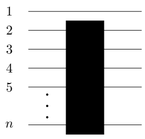
(53)

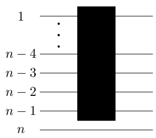
 $= \dfrac{1}{[n]_{q^{-1}}}$  $+ \dfrac{1-[n]_{q^{-1}}}{[n]_{q^{-1}}}$ 

(ii)

(54)

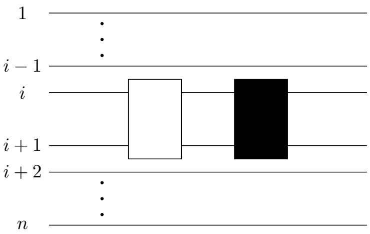
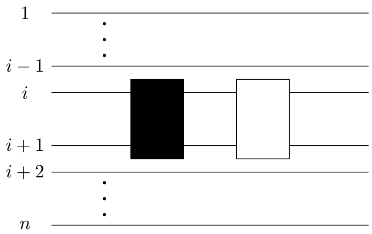
 $= \dfrac{1}{[n]_{q^{-1}}}$  $+ \dfrac{1-[n]_{q^{-1}}}{[n]_{q^{-1}}}$ 

Proof. **(i)** Multiply equation (46), on the left, by  and use Theorem 4.

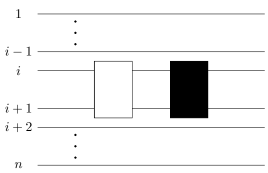
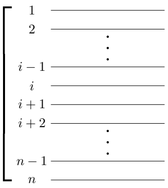
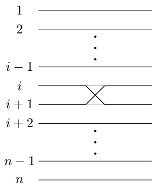
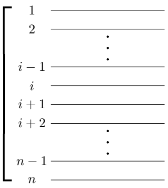
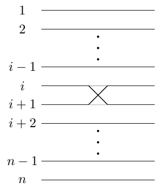
(ii) Multiply equation (52), on the left, by  and use Theorem 4. □

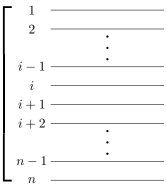
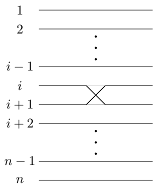
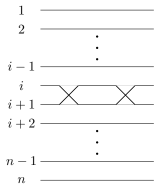
Proposition 1. *For $i \in 1, 2, \ldots, n-1$ we have that $A_{i,i+1}S_{i,i+1} = 0 = S_{i,i+1}A_{i,i+1}$.*

(55)

 $= 0 =$ 

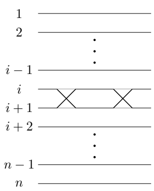
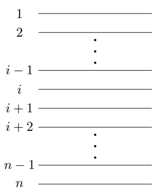
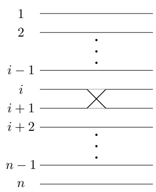
Proof.

 $= \left(\dfrac{1}{[2]_q!}\,\dfrac{1}{[2]_{q^{-1}}!}\right) \Bigg[$  $+$  $\Bigg]\Bigg[$  $- q^{-1}$  $\Bigg]$

$= \left(\dfrac{1}{[2]_q!}\,\dfrac{1}{[2]_{q^{-1}}!}\right) \Bigg[$  $+ (1-q^{-1})$  $- q^{-1}$  $\Bigg]$

$= 0.$

The last equality holds because, by item (a) of Definition 1, we have that

q^{-1}  $=$  $+ (1-q^{-1})$ . □

Corollary 5. *If a symmetrizer and an antisymmetrizer have at least two strings in common, then their product is zero.*

Proof. Suppose that the common strings to the symmetrizer $S_{k,\dots,m}$ and to the antisymmetrizer $A_{a,\dots,b}$ are j and $j+1$. Then,

$$S_{k,\dots,m}A_{a,\dots,b} = (S_{k,\dots,m}S_{j,j+1})(A_{j,j+1}A_{a,\dots,b}) = S_{k,\dots,m}(S_{j,j+1}A_{j,j+1})A_{a,\dots,b} = S_{k,\dots,m}(0)A_{a,\dots,b} = 0. \quad (56)$$

□

In [2], the author develops a graphical calculus corresponding to the contraction of indices of tensors. It is possible to extend this concept to the Hecke algebra case, but doing so requires a more detailed knowledge of the representation theory of the Hecke algebra. We hope to pursue this idea in a future paper.

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